# DEFORMATION OF LIE-ALGEBRAS AND LIE-ALGEBRAS OF DEFORMATIONS.

# APPLICATION TO THE STUDY OF HYPERSURFACE SINGULARITIES

by

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INTRODUCTION

This paper is a following up of some of the main ideas of [La-Pf], on which it depends notationally, as well as philosophically. As we show (loc.cit. (2.6) (iv),(v)) the study of local moduli for a scheme $X$ naturally leads to the study of flat families of Lie-algebras. In fact, if $\pi:\tilde{X}\to\underline{H}=\mathrm{Spec}(H)$ is a miniversal deformation of $X$, and if $\underline{V}$ is the kernel of the Kodaira-Spencer map associated to $\pi$, then the prorepresenting substratum $\underline{H}_0=\mathrm{Spec}(H_0)\subseteq\underline{H}$ is the complement of the support of $V\subseteq\mathrm{Der}_k(H)$, and

$$\mathcal{L} = V\underset{H}{\otimes}H_0.$$

is an $H_0$-Lie algebra defining a deformation of the Lie-algebra $L(X)=A^0(k,X;O_X)/A_1^0 = H^0(X,\theta_X)/A_1^0$ where $A_1^0$ is the Lie-ideal of those infinitesimal automorphisms of X that lifts to $A^0(H,\tilde{X};\theta_{\tilde{X}})$.
If $X=\mathrm{Spf}(k[[\underline{x}]]/(f))$ where $f\in k[\underline{x}]$ is an isolated hypersurface singularity, then putting $L(f)=L(X)$ we find (see §4 loc. cit.) that

$$L(f) = \mathrm{Der}_k(k[[\underline{x}]]/(f))/\mathrm{Der}_1$$

where $\mathrm{Der}_1$ is the Lie-ideal generated by the derivations of the form $E_{ij} = \frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial x_i}$, $i<j$. In particular

$$\dim L(f) = \tau(f) .$$

The family $\mathcal{L}$ defines, in a natural way a map:

$\ell:\underline{H}_0 \to$ moduli space of Lie-algebras of $\dim.\tau(f)$,

associating to the closed point $\underline{t}$ of $\underline{H}_0$ the class of the Lie-algebra $L(F_0(\underline{t}))$ where $F_0$ is the restriction of the miniversal family $F$ of $f$ to $\underline{H}_0$.

The purpose of this paper is the proof in a special case of a conjectural imbedding theorem, see §3, which states that for any isolated hypersurface singularities, $\ell$ is a rational locally injective map, except for some very special cases.

To make this statement precise we first have to recall a few facts from the moduli theory of Lie-algebras, some of which we have not been able to find in the literature. Moduli theory for Lie-algebras has, in one way or another, been studied by a large number of mathematicians, for a long time, see for example [C], [Mo], [Ri], [Ra], [V1], [Fi].

Identifying a Lie-algebra, up to base change, with the set of structural constants $\{c_{ij}^k\}$, one sees immediately that the set of isomorphism classes of n-dimensional Lie-algebras defined over a field k of characteristic $\neq 2$, may be identified with the set of orbits $\underline{L}_n = \underline{Lie}_n / Gl_n(k)$, where $\underline{Lie}_n$ is an affine algebraic subscheme of the space $\underline{A}_k^{\frac{1}{2}n^2(n-1)}$, of all systems of structural constants $\{c_{ij}^k\}$, $i<j$, $i,j,k=1,\dots,n$, defined by a set of quadratic equations deduced from the Jacobi identities, see §1, Ex.1.

Since $Gl_n(k)$ is reductive there exists in the category of schemes a categorical quotient of $\underline{Lie}_n$ by $Gl_n(k)$, see [M-F], (1.1). However since the action of $Gl_n(k)$ is not closed this quotient is not geometric and the set of closed points cannot, in general, be identified with the set of orbits $\underline{L}_n$.

The structure of $\underline{L}_n$ is, in fact, highly complex, and very little is known about it. To formulate and prove the imbedding theorem (3.2), we therefore have to work a little, developing the deformation theory and the local moduli of Lie-algebras along the lines of [La1] and [La-Pf]. We start by taking another look at the cohomology of Lie-algebras, amounting essentially to a reformulation of the classical Chevalley cohomology. This is the subject of §1.

With this reformulation of the cohomology at hand, we may carry over to the Lie-algebra case the obstruction calculus for the deformation functor of [La1] and many of the results of §1-§3 of [La-Pf]. In particular we obtain for any flat family of Lie-algebras a Kodaira Spencer map. Theorem (3.10) of [La-Pf] carries over to the Lie-algebra situation, giving useful formulas for the dimension of the rigidifying components of the base of a miniversal family of any Lie-algebra $\mathfrak{g}$. If $\mathfrak{g}$ is semi-simple we find, in particular, that the component of $\underline{\mathrm{Lie}}_n$ containing $\mathfrak{g}$ has dimension $n(n-1)$, and is a closure of an orbit of $\mathrm{Gl}_n(k)=\mathrm{Gl}(\mathfrak{g})$ isomorphic to $\mathrm{Gl}(\mathfrak{g})/\exp(\mathfrak{g}^{ad})$, see (2.2).

Now, the Kodaira-Spencer map associated to the family defined above, induces a commutative diagram for every closed point $\underline{t}\in\underline{H}_0$

$$\begin{array}{ccc} \mathrm{Der}(H_0) & \xrightarrow{\lambda} & A^1(H_0,\mathcal{L}:\mathcal{L}) \\ \downarrow & & \downarrow \\ t_{\underline{H}_0,t} & \xrightarrow[\lambda(\underline{t})]{} & A^1(k,L(\underline{t});L(\underline{t}) \end{array}$$

where $t_{\underline{H}_0,\underline{t}}$ is the tangent space of $\underline{H}_0$ at $\underline{t}$, and where we have put $L(\underline{t})=L(F_0(\underline{t}))$. Here $\lambda(\underline{t})$ is the tangent map of the, not yet well defined, morphism $\ell$. This done, we shall prove the "Imbedding theorem" (3.1) in the following form; see (3.2).

<u>Let $f$ be a quasihomogenous isolated plane curve singularity of the form $x^k+y^\ell$. Suppose the dimension of the prorepresenting substratum $H_0^{(f)}$ is $\geq 2$. Then there is an open dense subset $U$ of $\underline{H}_0(f)$ such that for every closed point $\underline{t}\in U$, the map $\lambda(\underline{t})$ is injective.</u>

In [M-Y] and [Y], Mather and Yau consider the correspondences

$$f \to k[\underline{x}]/(f, \frac{\partial f}{\partial x_i}) = A^1(f)$$

$$f \to \mathrm{Der}_k(A^1(f)) \ .$$

They prove that $A^1(f)$, as a k-algebra, characterizes the singularity $f$, and they may, for low dimensions, prove that $\mathrm{Der}_k(A^1(f))$ is a solvable Lie-algebra.

Even for very simple singularities like $E_6$, our $L(f)$ is however different from $\mathrm{Der}(A^1(f))$, and we don't see any immediate relationship between these two invariants.

Notice also that for general singularities $A^1(k,X;O_X)$ is not an algebra.

At the end of §3 we give some examples of special cases where (3.1) fails. In §4 we finally include a list of invariants like the generalized Cartan matrix for some simple singularities. The last list should be compared with that of [Y].

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## §1. Cohomology of Lie-algebras

For the purpose of studying deformations of Lie-algebras we need a cohomology theory and an obstruction calculus, see [La1]. There is such a cohomology theory, due to Chevalley [C] and one knows how to define the obstructions we need, see [Ri], [Ra]. However we shall, never the less, in this paragraph, define "another" cohomology theory, that fits more naturally into our development of the deformation theory, as described in [La1]. Of course we shall prove that the new cohomology and the old one coincide modulo a change of degree, and apart from the first few groups.

Let us first recall the Chevalley-Eilenberg-MacLane cohomology of a Lie-algebra $\mathfrak{g}$ defined on a field k, see [E-ML]. Consider the functor:

$$H^0(\mathfrak{g};-):\mathfrak{g}\text{-}\underline{\text{mod}} \to k\text{-}\underline{\text{mod}}$$

defined by $H^0(\mathfrak{g},M)=M^{\mathfrak{g}}=\{m\in M \mid \forall\delta\in\mathfrak{g},\ \delta(m)=0\}$. Let $H^i(\mathfrak{g},M)$ be the "i-th derived functor", see [E-ML], applied to the $\mathfrak{g}$-modul M. then $H^i(\mathfrak{g},M)$ $i>0$ is the <u>cohomology of $\mathfrak{g}$ with values in M</u>. As usual there is an exact functor of complexes

$$M \to C^{\cdot}(\mathfrak{g},M)$$

with, in this case,

$$C^p(\mathfrak{g},M) = \mathrm{Hom}_k(\overset{p}{\Lambda}\mathfrak{g},M)$$

and differential

$$d:C^p(\mathfrak{g},M) \to C^{p+1}(\mathfrak{g},M)$$

defined by:

$$d(f)(\gamma_0\wedge\ldots\wedge\gamma_p) = \sum_{i=0}^{p}(-1)^i\gamma_i f(\gamma_0\wedge\ldots\wedge\hat{\gamma}_i\wedge\ldots\wedge\gamma_p)$$

$$+ \sum_{0<i<j<p}(-1)^{i+j}f([\gamma_i,\gamma_j]\wedge\gamma_0\wedge\ldots\wedge\hat{\gamma}_i\wedge\ldots\wedge\hat{\gamma}_j\wedge\ldots\wedge\gamma_p)$$

such that:

$$H^p(\mathfrak{g},M) \simeq H^p(C^\cdot(\mathfrak{g},M) \qquad p>0 .$$

Now, let $S$ be any commutative ring with unit. We may consider the category S-<u>lie</u> of Lie-algebras defined over $S$ and the full subcategory

<u>free</u> $\subseteq$ S-<u>lie</u>

of the free S-Lie-algebras, see [J]. Given any S-Lie-algebra $\mathfrak{g}$ and an S-module $M$ a structure of $\mathfrak{g}$-module on $M$ is, of course, nothing but a homomorphism of S-Lie-algebras

$$\rho:\mathfrak{g}\to \mathrm{End}_S(M) .$$

We shall denote by

$\mathfrak{g}$-<u>mod</u>

the category of $\mathfrak{g}$-modules, supressing the reference to the base ring $S$.

In the particular case where the S-Lie algebra $\mathfrak{g}$ is free as an S-module, we may for any $\mathfrak{g}$-module $M$ extend the definition of the complex $C^\cdot(\mathfrak{g}, M)$ above, simply by putting

$$C^p(\mathfrak{g},M) = \mathrm{Hom}_S(\overset{p}{\underset{S}{\Lambda}}\mathfrak{g},M).$$

As above we denote by

$$H^\cdot(\mathfrak{g},M)$$

the resulting chomology.

Associated to any S-Lie-algebra $\mathfrak{g}$ there is the category

$\underline{\text{free}}/\mathfrak{g}$

of all morphisms of S-Lie-algebras $\mu:F\to\mathfrak{g}$ where $F$ is a free S-Lie algebra, a morphism $\phi$ of $\underline{\text{free}}/\mathfrak{g}$ being a commutative diagram

$$\begin{tikzcd} F_1 \arrow[rr, "\phi"] \arrow[dr, "\mu_1"'] & & F_2 \arrow[dl, "\mu_2"] \\ & \mathfrak{g} & \end{tikzcd}$$

Consider the contravariant functor $\mathfrak{g}\to\mathrm{Der}_S(\mathfrak{g},M)$ defined on the category S-$\underline{\text{Lie}}$ by:

$$\mathrm{Der}_S(\mathfrak{g},M) = \{D\in\mathrm{Hom}_S(\mathfrak{g},M)\,|\,D([\delta_1,\delta_2]) = \delta_1 D(\delta_2)-\delta_2 D(\delta_1)\}$$

Obviously this functor induces a functor

$$\mathrm{Der}_S(-,M):(\underline{\text{free}}/\mathfrak{g})^{o} \to S\text{-}\underline{\text{mod}}$$

and we define, just like in [La1], chap.2, the cohomology

$$A^i(S,\mathfrak{g};M) = \varprojlim_{(\underline{\text{free}}/\mathfrak{g})^o}{}^{(i)} \mathrm{Der}_S(-,M).$$

Notice that there is an S-module homomorphism

$$i:M\to\mathrm{Der}_S(\mathfrak{g},M)$$

defined by $i(m)(\delta)=\delta m$, $m\in M$, $\delta\in\mathfrak{g}$. Notice also that if $\pi:R\to S$ is a homomorphism of commutative rings such that $(\ker\pi)^2=0$, and if there is given a diagram of morphisms of Lie-algebras

$$\begin{tikzcd} \mathfrak{g}_1' \arrow[r, shift left, "\phi_1"] \arrow[r, shift right, "\phi_2"'] \arrow[d, "\nu_1"] & \mathfrak{g}_2' \arrow[d, "\nu_2"] \\ \mathfrak{g}_1 \arrow[r, "\phi"'] & \mathfrak{g}_2 \end{tikzcd}$$

such that $\mathfrak{g}_i$ are flat S-Lie-algebras and $\mathfrak{g}_i'$ are flat R-Lie-algebras, the vertical morphisms $\nu_i$ being tensorization by S on R, then if $\phi_1$ and $\phi_2$ are both liftings of $\phi$ we find that the R-homomorphism $\phi_1-\phi_2$ decomposes into $j\cdot D\cdot\nu_1$ where $D:\mathfrak{g}_1 \to \mathfrak{g}_2 \underset{S}{\otimes} \ker\pi$ is a derivation and $j:\mathfrak{g}_2 \underset{S}{\otimes} \ker\pi = \mathfrak{g}_2' \underset{R}{\otimes} \ker\pi \to \mathfrak{g}_2'$ is the obvious injection. With this done we may copy the procedure of [La1] and obtain an obstruction calculus for the deformation functor of any S-Lie-algebra (see [La1], chap. 4). Before we sketch the results let's prove the following

<u>Theorem (1.1)</u>. Given any S-Lie-algebra $\mathfrak{g}$ which is free as an S-module, and any $\mathfrak{g}$-module M, then

(i) there exists an exact sequence

$$0 \to H^0(\mathfrak{g},M) \to M \xrightarrow{i} A^0(S,\mathfrak{g};M) \to H^1(\mathfrak{g},M) \to 0$$

(ii) there are isomorphisms

$$\theta_i : A^i(S,\mathfrak{g},M) \xrightarrow{\sim} H^{i+1}(\mathfrak{g},M) \quad i\geq 1$$

<u>Proof</u>. (i) is a simple consequence of the definition of $H^1(\mathfrak{g},M)$ using the standard complex $C^\cdot(\mathfrak{g},M)$. Recall that $H^i(\mathfrak{g},-)=0$ for $i\geq 2$, if $\mathfrak{g}$ is a free-S-Lie-algebra, and consider the functor

$$C^\cdot(-,M):(\underline{\text{free}}/\mathfrak{g})^o \to \underline{\text{Compl}}.\text{ of }\underline{S\text{-modules}}$$

Since for every object $\delta:F\to\mathfrak{g}$ in $(\underline{\text{free}}_S/\mathfrak{g})^o$, $C^\cdot(\delta,M)=\mathrm{Hom}_S(\overset{P}{\underset{S}{\Lambda}}F,M)$ it is easy to see that

$$\varprojlim_{(\underline{\text{free}}/\mathfrak{g})^o}{}^{(P)} C^q(-,M) = \mathrm{Hom}_S(\varinjlim_{(\underline{\text{free}}/\mathfrak{g})}{}_{(p)} \overset{q}{\Lambda}\mu,M)$$

where $\mu:(\underline{\text{free}}/\mathfrak{g}) \to S\text{-}\underline{\text{mod}}$ is the forgetfull functor $\mu(\delta)=F$. Using the Leray spectral sequence for $\mu$ (see [La1], chap. 2) knowing that for $\delta:F\to\mathfrak{g}$ surjective, the complex of S-modules

$$* \qquad \underset{\partial_{n+1}}{\overset{\partial_0}{\rightrightarrows}} F\times_{\mathfrak{g}}\cdots\times_{\mathfrak{g}} F \underset{\partial_n}{\overset{\partial_0}{\rightrightarrows}} \cdots \underset{\partial_2}{\overset{\partial_0}{\rightrightarrows}} F\times_{\mathfrak{g}} F \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} F \xrightarrow{\partial} \mathfrak{g} \to 0$$

is acyclic, we prove that $\varinjlim_{(\underline{\text{free}}/\mathfrak{g})}{}^{(p)} \mu=0$ if $P>1$ and $\varinjlim_{(\underline{\text{free}}/\mathfrak{g})} \mu = \mathfrak{g}$.

Notice that the functor $\overset{q}{\Lambda}\mu$ is not additive. Never the less we easily generalize the above, and we get

$$** \qquad \varinjlim_{(\underline{\text{free}}/\mathfrak{g})}{}^{(p)} \overset{q}{\Lambda}\mu = \begin{cases} 0 & p>1 \\ \overset{q}{\Lambda} & p=0 \end{cases} .$$

In fact, since $\mathfrak{g}$ is S-free we may find an S-linear section $h:\mathfrak{g}\to F$ of $\delta$. Define

$$h:\underbrace{F\times_{\mathfrak{g}}\cdots\times_{\mathfrak{g}}F}_{n+1} \to \underbrace{F\times_{\mathfrak{g}}\cdots\times_{\mathfrak{g}}F}_{n+2}$$

by $h(f_0,f_1,\ldots,f_n)=(f_0,f_1,\ldots,f_n,h(f_n))$. Since $\partial_i(f_0,\ldots,f_i,\ldots f_n)=(f_0,\ldots\hat{f}_i,\ldots,f_n)$ we find that

$$h\partial_i = \partial_i h \quad \text{for} \quad i<n$$

$$\partial_{n+1}h = \text{id} .$$

Applying $\overset{q}{\Lambda}_S$ to $*$ we observe that $\overset{q}{\Lambda}_S h$ defines a homotopy, or a retraction, proving that

$$\rightrightarrows \overset{q}{\Lambda}(F\times_{\mathfrak{g}}\cdots\times_{\mathfrak{g}}F) \rightrightarrows \cdots \rightrightarrows \overset{q}{\Lambda}(F\times_{\mathfrak{g}}F) \rightrightarrows \overset{q}{\Lambda}F \to \overset{q}{\Lambda}\mathfrak{g} \to 0$$

in acyclic. Now, since $\mu:(\underline{\text{free}}/\mathfrak{g}) \to S\text{-}\underline{\text{mod}}$ is the forgetfull functor, we know, see [La 1], chap. 2, that there exists a spectral sequence given by

$$E_2^{r,s} = H^r(\underset{(\underline{\text{free}}/F.)}{\varinjlim}{}_{(s)} \overset{q}{\Lambda}\mu) \Rightarrow \underset{(\underline{\text{free}}/\mathfrak{g})}{\varinjlim}{}_{(r+s)} \overset{q}{\Lambda}\mu$$

where $F_i = \underbrace{F \underset{\mathfrak{g}}{\times} \cdots \underset{\mathfrak{g}}{\times} F}_{i+1}$ and $F.$ is the complex $*$. Since $F_0$ is free, we obtain in exactly the same way as in ([La1], (2.1.5))

$$E_2^{0,1} = 0 .$$

Since moreover:

$$E_2^{1,0} = H^1(\overset{q}{\Lambda}F.) = 0$$

we conclude

$$\underset{(\underline{\text{free}}/\mathfrak{g})}{\varinjlim}{}_{(1)} \overset{q}{\Lambda}\mu = 0 .$$

From this we easily obtain $**$. But $**$ implies that

$$\underset{(\underline{\text{free}}/\mathfrak{g})^{o}}{\varprojlim}{}^{(p)} C^q(-,M) = \begin{cases} C^q(\ ,M) & p=0 \\ 0 & p\geq 1 \end{cases} .$$

Consider now the spectral sequences associated to the double complex

$$C^{\cdot}((\underline{\text{free}}/\mathfrak{g})^{o}; C_1^{\cdot}(-,M))$$

where we have put

$$C_1^{\cdot}(-,M) = C^{\cdot+1}(-,M)$$

and where $C^{\cdot}((\underline{\text{free}}/\mathfrak{g})^{o},-)$ is the resolving complex for the functor $\underset{(\underline{\text{free}}/\mathfrak{g})^{o}}{\varprojlim}$. Since for any free S-Lie-algebra $F$

$$H^q(C_1^{\cdot}(F,M)) = \begin{cases} \mathrm{Der}_S(F,M) & q=0 \\ 0 & q\geq 1 \end{cases}$$

we find that the first spectral sequence of the double complex is given by:

$$'E_2^{p,q} = H^p(C^\cdot((\underline{\text{free}}/\mathfrak{g})^o; H^q(C_1^\cdot(-,M)) = \begin{cases} A^p(S, \;;M) & q=0 \\ 0 & q\geq 1 \end{cases}.$$

From what we just saw above we deduce that the second spectral sequence is given by:

$$''E_2^{p,q} = H^p(\varprojlim^{(q)}_{(\underline{\text{free}}/\mathfrak{g})^o} C_1^\cdot(-,M)) = \begin{cases} H^p(C_1^\cdot( \;,M)) & q=0 \\ 0 & q\geq 1 \end{cases}.$$

This ends the proof of the theorem. Q.E.D.

Now, using the Leray-spectral sequence, see (2,1.3) and the subsequent Remark 1 of [La1], we may easily compute the first few cohomology groups of $\mathfrak{g}$, assuming of course that $\mathfrak{g}$ is given in terms of its structural constants. In fact, let $\{x_i\}$, $i=1,\dots,n$ be a basis of $\mathfrak{g}$ and let for $1\leq i, j\leq n$,

$$[x_i,x_j] = \sum_{k=1}^n c_{ij}^k x_k, \quad c_{ij}^k \in S.$$

Consider the free Lie-algebra $F$ generated by the symbols $\bar{x}_i$ and let $j:F\to\mathfrak{g}$ be the Lie-algebra morphism defined by $\rho(\bar{x}_i)=x_i$. Then $J=\ker j$ is generated as ideal of $F$ by the elements

$$f_{ij} = [\bar{x}_i,\bar{x}_j] - \sum_{k=1}^n c_{ij}^k \bar{x}_k, \quad 1\leq i,j\leq n$$

with "linear" relations $I \subseteq \coprod_{1\leq i,j\leq n} F_{ij}$, $F_{ij}=F$, containing the elements:

$$\begin{aligned} r_{i,j,k} &= [f_{ij},\bar{x}_k] + [f_{jk},\bar{x}_i] + [f_{ki},\bar{x}_j] \\ &+ \sum_{l=1}^n c_{ij}^l f_{lk} + \sum_{l=1}^n c_{jk}^l f_{li} + \sum_{l=1}^n c_{ki}^l f_{lj}. \end{aligned}$$

Proposition (1.2). With the notations above we find:

$$A^1(S,\mathfrak{g};M) = \mathrm{Hom}_F(J,M)/\mathrm{Der}$$
$$A^2(S,\mathfrak{g};M) = \mathrm{Hom}_F(I,M)/\mathrm{Der}$$

When $S=k$ is a field one immediately checks that the isomorphisms

(i) $A^1(k,\mathfrak{g};M) \overset{\theta_1}{\tilde{\to}} H^2(\mathfrak{g},M)$

(ii) $A^2(k,\mathfrak{g};M) \overset{\theta_2}{\tilde{\to}} H^3(\mathfrak{g},M)$

of (1.1) is given as follows.

(i) Let $\phi \in H^2(\mathfrak{g},M)$ be given in terms of a cocycle $f \in \mathrm{Hom}_k(\mathfrak{g}\wedge\mathfrak{g},M)$, then the map

$$f_{ij} \to f(x_i \wedge x_j)$$

extends to an F-linear map $J \to M$ defining an element $\xi \in A^1(k,\mathfrak{g};M)$, such that $\theta_1(\xi)=\phi$.

(ii) Let $\phi \in H^3(\mathfrak{g},M)$ be given in terms of a cocycle $r \in \mathrm{Hom}_k(\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g},M)$, then the map

$$r_{ijk} \to r(x_i \wedge x_j \wedge x_k)$$

extends to an F-linear map $I \to M$ defining an element $\rho \in A^2(k,\mathfrak{g};M)$, such that $\theta_2(\rho)=\phi$.

It is now easy to construct the obstructions we need for the "obstruction calculus". In fact if $\pi: R \to S$ is a surjective homomorphism of commutative rings with unit, such that $\ker\pi^2=0$ and if $\mathfrak{g}$ is an S-Lie-algebra, flat over S, given, as above, in terms of its structural constants $\{c_{ij}^k\}$, any lifting of $\mathfrak{g}$ to R must necessarily have structural constants $\bar{c}_{ij}^k \in R$ such that $\pi(\bar{c}_{ij}^k)=c_{ij}^k$. To see whether there is such a lifting or not, we pick $\bar{c}_{ij}^k \in R$ satisfying $\pi(\bar{c}_{ij}^k)=c_{ij}^k$ and consider the map

$$\begin{aligned} r_{ijk} \to & [[\bar{x}_i,\bar{x}_j],\bar{x}_k] + [[\bar{x}_j,\bar{x}_k],\bar{x}_i] + [[\bar{x}_k,\bar{x}_i],\bar{x}_j] \\ &= \sum_m \Big(\sum_l \bar{c}_{ij}^l \bar{c}_{lk}^m + \bar{c}_{jk}^l \bar{c}_{li}^m + \bar{c}_{ki}^l \bar{c}_{lj}^m\Big) x_k \in \mathfrak{g} \underset{S}{\otimes} \ker\pi \end{aligned}$$

It extends to an F-linear map

$$I \to \mathfrak{g} \underset{S}{\otimes} \ker\pi$$

defining an element $\sigma(\pi,\mathfrak{g}) \in A^2(S,\mathfrak{g};\mathfrak{g} \underset{S}{\otimes} \ker\pi)$.

<u>Proposition (1.3)</u>. With the notations above, there exists an obstruction $\sigma(\pi,\mathfrak{g}) \in A^2(S,\mathfrak{g};\mathfrak{g} \underset{S}{\otimes} \ker\pi)$ such that $\sigma(\pi,\mathfrak{g})=0$ if and only if there exists a lifting of $\mathfrak{g}$ to R, in which case the set of isomorphism classes of liftings is a principal homogenous space (torsor) over $A^1(S,\mathfrak{g};\mathfrak{g} \underset{S}{\otimes} \ker\pi)$.

<u>Proof</u>. This follows immediately from the definition of $\sigma(\pi,\mathfrak{g})$ above, together with (1.2). Notice that we may also copy the proof from that of ((2.2.5) [La1]) Q.E.D.

Copying the definition of cup-product from ((5.1.5) [La1]) we find a map,

$$\vee : A^1(k,\mathfrak{g};\mathfrak{g}) \to A^2(k,\mathfrak{g};\mathfrak{g})$$

defined as follows: Let $\xi \in A^1(k,\mathfrak{g};\mathfrak{g})$ be given in terms of an F-linear map $k: J \to \mathfrak{g}$ such that $h(f_{ij}) = \sum_{k=1}^{n} h_{ij}^{k} x_k$, then $v\xi \in A^2(k,\mathfrak{g};\mathfrak{g})$ is given in terms of the F-linear map $vh: I \to \mathfrak{g}$ defined by:

$$\begin{aligned} vh(r_{ijk}) &= h\Big(\sum_{l=1}^{n} h_{ij}^{l}[\bar{x}_l,\bar{x}_k] + \sum_{l=1}^{n} h_{jk}^{l}[\bar{x}_l,\bar{x}_i] + \sum_{l=1}^{n} h_{ki}^{l}[\bar{x}_l,\bar{x}_j] \\ &\quad + \sum_{l,m} c_{ij}^{l} h_{lk}^{m}\bar{x}_m + \sum_{l,m} c_{jk}^{l} h_{li}^{m}\bar{x}_m + \sum_{l,m} c_{ki}^{l} h_{lj}^{m}\bar{x}_m\Big) \\ &= \sum_{l,m} (h_{ij}^{l}h_{lk}^{m} + h_{jk}^{l}h_{li}^{m} + h_{ki}^{l}h_{lj}^{m}) x_m \end{aligned}$$

which is nothing but the map $\Theta$ of Rim, [Ri].

<u>Example 1</u>. If $\mathfrak{g}$ is the abelian $n$ dimensional Lie-algebra then

$$A^1(k,\mathfrak{g};\mathfrak{g}) \simeq \mathfrak{g}^{\frac{n(n-1)}{2}} \simeq k^{\frac{n^2(n-1)}{2}}$$

$$A^2(k,\mathfrak{g};\mathfrak{g}) \simeq \mathfrak{g}^{N} \simeq k^{n\cdot N}$$

where $N = \#\{\text{generators } r_{ijk}\}$. The map

$$v: A^1(k,\mathfrak{g};\mathfrak{g}) \to A^2(k,\mathfrak{g};\mathfrak{g})$$

is the quadratic map

$$v\{h_{ij}^{k}\}_{i,j,k} = \{\sum_{l} h_{ij}^{l}h_{lk}^{m} + h_{jk}^{l}h_{li}^{m} + h_{ki}^{l}h_{lj}^{m}\}_{i,j,k,m}$$

Notice the similarity with the quadratic forms in the defining equations of the affine subscheme $\underline{\mathrm{Lie}}_n \subseteq A^{\frac{n^2(n-1)}{2}}$ deduced from the Jacobi identities. They are easily seen to be

$$\sum_{l=1}^{n} (c_{ij}^{l}c_{lk}^{m} + c_{jk}^{l}c_{li}^{m} + c_{ki}^{l}c_{lj}^{m}) = 0 \qquad i,j,k,m$$

where we as usual have assumed char $k \neq 2$ and $c^l_{ij} = -c^l_{ji}$ for all $i,j,l$. This is, as we shall see a particular case of a general result (2.1) about the structure of the formal moduli of any Lie-algebra.

Example 2. If $\mathfrak{g}$ is semisimple it follows from (1.2) that $A^1(k,\mathfrak{g};\mathfrak{g})=0$ which is a classical result. In fact $H^i(\mathfrak{g},\mathfrak{g})=0$ for all $i>1$, and compare (1.1).

## §2. Deformations of Lie-algebras, and local moduli.

As above we denote by $k$ a fixed field, by $R, S$, etc. commutative $k$-algebras. We shall also, as usual, denote by $\underline{1}$ the category of local artinian $k$-algebras with residue field $k$. Given any $k$-Lie-algebra $\mathfrak{g}$ the deformation functor

$$\mathrm{Def}_{\mathfrak{g}} : \underline{1} \to \underline{\mathrm{Sets}}$$

is defined by

$$\mathrm{Def}\ (S) = \{\eta : \mathfrak{g}_S \otimes_S k \xrightarrow{\simeq} \mathfrak{g} \mid \text{where } \mathfrak{g}_S \text{ is an } S\text{-Lie-algebra free as } S\text{-module}\}/\sim$$

where $\sim$ is the equivalence relation defined by: $\eta \sim \eta'$ if there exists a commutative diagram

$$\begin{tikzcd} \mathfrak{g}_S \arrow[rr, "\chi", "\sim"'] \arrow[dr, "\eta"'] & & \mathfrak{g}'_S \arrow[dl, "\eta'"] \\ & \mathfrak{g} & \end{tikzcd}$$

where $\chi$ is an isomorphism of S-Lie-algebras. Now, using the obstructions $\sigma(\pi,\mathfrak{g}_S)$ and the Proposition (1.3) we may proceed exactly as in the proof of (4.2.4) of [La1] to obtain the following apparently well known result, see [Ra], [Fi].

Theorem (2.1). Suppose $\mathfrak{g}$ is of finite dimension and put $A^i = A^i(k,\mathfrak{g};\mathfrak{g})$. Then there exists a morphism of complete local k-algebras

$$\sigma : \mathrm{Sym}_k(A^{2*})^\wedge \to \mathrm{Sym}_k(A^{1*})^\wedge$$

such that putting $T^i = \mathrm{Sym}(A^{i*})^\wedge$,

$$H^\wedge = T^1 \underset{T^2}{\otimes} k$$

is a prorepresenting hull for the functor $\mathrm{Def}_{\mathfrak{g}}$. Moreover $\sigma$ maps the maximal ideal $\underline{m}_{T^2}$ of $T^2$ into the square $\underline{m}_{T^1}^2$ of the maximal ideal of $T^1$. The dual of the resulting map $\underline{m}_{T^2}/\underline{m}_{T^2}^2 \to \underline{m}_{T^1}^2/\underline{m}_{T^1}^3$ is the cup-product

$$\vee : A^1 \underset{\mathrm{Sym}}{\otimes} A^1 \to A^2$$

deduced from the quadratic map $\vee : A^1 \to A^2$ of §1. In particular the tangent space of the formal moduli $\mathrm{Spf}(H^\wedge)$ of $\mathfrak{g}$ is isomorphic to $A^1(k,\mathfrak{g};\mathfrak{g})$.

Example 3. It follows from Example 1 and the above theorem that the completion of $\underline{\mathrm{Lie}}_n$ at the origin is isomorphic to the formal moduli $\underline{H}^\wedge(k^n)$ of the abelian Lie-algebra $k^n$. Moreover the $(\mathrm{Lie}_n)$-Lie-algebra $\mathcal{L}_n$ with

$$\mathrm{Lie}_n = k[C_{ij}^k]_{i<j}/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generated by the quadratic forms of Example 1, is defined by the structural constants

$$\{\tilde{C}_{ij}^k\}$$

where $\tilde{C}^k_{ij}$ is the class of $C^k_{ij}$ in $\mathrm{Lie}_n$. $\mathcal{L}_n$ is therefore an "algebraization" of the universal formal family defined on $H^\wedge(k^n)$.

Given any k-Lie-algebra $\mathfrak{g}$, there exists by (2.1) a formal versal family, i.e. an $H^\wedge$-Lie-algebra $\mathfrak{g}^\wedge$, flat as $H^\wedge$-module and such that $\mathfrak{g}^\wedge \underset{H}{\otimes} k \simeq \mathfrak{g}$. Pick any basis $\{x_i^\wedge\}_{i=1}^n$ of $\mathfrak{g}^\wedge$ as free $H^\wedge$-module and consider the corresponding structural constants $\{\hat{C}^k_{ij}\}$. Consider the (finitely generated) k-subalgebra $H$ of $H^\wedge$ generated by the $\hat{C}^k_{ij}$. Since by construction $H^\wedge/\underline{m}^2$, where $\underline{m}$ is the maximal ideal, must be generated as k-algebra by the images of $\hat{C}^k_{ij}$, it follows readily that the completion of $H$ w.r.t. the ideal $\underline{m} \cap H$, is $H^\wedge$. Let $\tilde{\mathfrak{g}}$ be the H-Lie-algebra defined by the structural constants $\hat{C}^k_{ij}$, then $(H, \tilde{\mathfrak{g}})$ is an algebraization of <u>the</u> formal versal family $(H^\wedge, \hat{\mathfrak{g}})$. Thus we have proved the following,

<u>Lemma (2.2)</u>. For every k-Lie-algebra $\mathfrak{g}$ there exists an algebraization of the formal versal family $(H^\wedge, \hat{\mathfrak{g}})$.

Compare (2.2) to the condition $(A_1)$ of §3 [La-Pf].

We shall now apply the technique of [La-Pf] §1-3 to the study of Lie-algebras.

First we have to introduce the <u>Kodaira-Spencer</u> map of an S-Lie-algebra flat as an S-module, $\tilde{\mathfrak{g}}$. Following, word for word, the construction of the Kodaira- Spencer map of §3, loc.cit. we obtain an S-linear map

$$g : \mathrm{Der}_k(S) \to A^1(S, \tilde{\mathfrak{g}}; \tilde{\mathfrak{g}})$$

given explicitly by:

Let $\tilde{\mathfrak{g}}$ be given in terms of its structural constants $\tilde{C}^k_{ij} \in S$, and put

$$F_{ij} = [\bar{x}_i, \bar{x}_j] - \sum_{k=1}^{n} \tilde{C}^k_{ij} \bar{x}_k \in \tilde{J}$$

where $\tilde{j}:\tilde{F} \to \tilde{\mathfrak{g}}$ is a surjective morphism of a free S-Lie-algebra $\tilde{F}$ onto $\tilde{\mathfrak{g}}$, mapping the generators $\bar{x}_i$ onto the basis $x_i$ of $\tilde{\mathfrak{g}}$ and $\tilde{J} = \ker \tilde{j}$. We know by (1.2) that $A^1(S,\tilde{\mathfrak{g}};\tilde{\mathfrak{g}})$ is a quotient of $\mathrm{Hom}_{\tilde{F}}(\tilde{J};\tilde{\mathfrak{g}})$ and that $F_{ij}$ generate $\tilde{J}$. Let $D \in \mathrm{Der}_k(S)$, then $g(D)$ is the element of $A^1(S,\tilde{\mathfrak{g}};\tilde{\mathfrak{g}})$ determined by the homomorphism $\tilde{J} \to \tilde{\mathfrak{g}}$ given by $F_{ij} \mapsto D(F_{ij}) = -\sum_{k=1}^{n} D(\tilde{C}^k_{ij}) x_k$. This map is, by construction, such that for any closed point $\underline{s} \in \underline{S}$ if $\tilde{\mathfrak{g}}(s)$ is the fiber of $\tilde{\mathfrak{g}}$ at $\underline{s}$, the following diagram commute,

$$\begin{array}{ccc} \mathrm{Der}_k(S) & \xrightarrow{g} & A^1(S,\tilde{\mathfrak{g}};\tilde{\mathfrak{g}}) \\ \downarrow & & \downarrow \\ t_{\underline{S},\underline{s}} & \xrightarrow[g(\underline{s})]{} & A^1(k,\tilde{\mathfrak{g}}(\underline{s});\tilde{\mathfrak{g}}(\underline{s})) \end{array}$$

where $g(\underline{s})$ is the canonical map given by the formal family $(S^{\wedge}_{\underline{s}}, \tilde{\mathfrak{g}} \otimes_S S^{\wedge}_{\underline{s}})$. If $S=H$ is an algebraization of the formal moduli of $\mathfrak{g}$ $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ is the corresponding versal deformation, then as in (3.3) and (3.5) loc.cit. we deduce that the kernel $V$ of $g$ is a sub k-Lie-algebra of $\mathrm{Der}_k(H)$ and that

$$V \underset{H}{\otimes} k \simeq \mathrm{Der}(\mathfrak{g})/\mathrm{Der}_1$$

where $\mathrm{Der}_1$ is the image of $\mathrm{Der}_H(\tilde{\mathfrak{g}})$ in $\mathrm{Der}(\mathfrak{g})$. If the generic fiber of every component of $\underline{H}$ is rigid and complete, we find that $\mathrm{Der}_1 = \mathfrak{g}^{ad}$. In general we may only conclude that $\mathfrak{g}^{ad} \subseteq \mathrm{Der}_1$.

Theorem (2.3). Let $\mathfrak{g}$ be any k-Lie-algebra. Then the algebraization $(H,\tilde{\mathfrak{g}})$ of the formal versal family $(H^{\wedge},\mathfrak{g}^{\wedge})$ is locally formally versal in the sense of (3.6) [La-Pf].

Proof. We have to prove that there exists an open neighbourhood $\underline{U}$ of the base point such that for every closed point $\underline{t}\in\underline{U}$, the map $g(\underline{t})$ in the diagram

$$\begin{array}{ccc} \mathrm{Der}_k(H) & \xrightarrow{g} & A^1(H,\tilde{\mathfrak{g}};\tilde{\mathfrak{g}}) \\ \downarrow & & \downarrow \\ t_{\underline{H},t} & \xrightarrow[g(\underline{t})]{} & A^1(k,\tilde{\mathfrak{g}}(\underline{t});\tilde{\mathfrak{g}}(\underline{t})) \end{array}$$

is surjective.

Now, since we know there exists a map,

$$\alpha: \underline{H} \to \underline{\mathrm{Lie}}_n$$

such that the pull back of $\mathcal{L}_n$ is $\tilde{\mathfrak{g}}$, and since $(H^{\wedge},\mathfrak{g}^{\wedge})$ is formally versal, we find using M. Artins approximation theorem, an etale neighbourhood $\underline{E}$ of $\mathfrak{g}\in\underline{\mathrm{Lie}}_n$ and a diagram of morphisms

$$\begin{array}{ccc} \underline{E} & \xrightarrow{\beta} & \underline{H} \\ i\downarrow & \swarrow\alpha & \\ \underline{\mathrm{Lie}}_n & & \end{array}$$

such that $i^*(\mathcal{L}_n)\simeq\beta^*\tilde{\mathfrak{g}}$. Let $\underline{U}=\mathrm{im}\beta$ and let $\underline{t}\in\underline{U}$ be a closed point. Pick a point $\underline{t}'\in\underline{E}$ s.t. $\beta(\underline{t}')=\underline{t}$. Consider the Lie-algebra $\tilde{\mathfrak{g}}(\underline{t})$, and its formal moduli $H^{\wedge}(\underline{t})$. By definition, we know that $H^{\wedge}(\underline{t})/\underline{m}^2 \simeq k\oplus A^1(k,\tilde{\mathfrak{g}}(\underline{t});\tilde{\mathfrak{g}}(\underline{t}))^*$, where $\underline{m}$ is the maximal ideal of $H^{\wedge}(\underline{t})$. Denote by $\tilde{\mathfrak{g}}(\underline{t})_2^{\wedge}$ the "universal" deformation of $\tilde{\mathfrak{g}}(t)$ to $H^{\wedge}(\underline{t})/\underline{m}^2$. There is a morphism $\gamma: \mathrm{Spec}(H^{\wedge}(\underline{t})/\underline{m}^2) \to \underline{\mathrm{Lie}}_n$ such that $\gamma^*\mathcal{L}_n\simeq\tilde{\mathfrak{g}}(t)_2^{\wedge}$. Since $i$ is etale there is a morphism $\delta: \mathrm{Spec}(H^{\wedge}(t)/\underline{m}^2) \to \underline{E}$ such that the following diagram commutes:

$$\begin{tikzcd} & & \underline{E} \arrow[r, "\beta"] \arrow[d] & \underline{H} \arrow[dl, "\alpha"] \\ \mathrm{Spec}(k(\underline{t}')) \arrow[r] & \mathrm{Spec}(H^{\wedge}(t)/\underline{m}^2) \arrow[ur, "\delta"] \arrow[r, "\gamma"'] & \underline{\mathrm{Lie}}_n & \end{tikzcd}$$

But this implies that $\delta^*\beta^*\tilde{\mathfrak{g}} \simeq \tilde{\mathfrak{g}}(\underline{t})^{\wedge}_2$, and in particular, the tangent map induced by $\beta\circ\delta$,

$$A^1(k(\underline{t}),\tilde{\mathfrak{g}}(\underline{t});\tilde{\mathfrak{g}}(\underline{t})) \to t_{\underline{H},\underline{t}}$$

must be injective. But this map must be a section of the functorial map

$$g(\underline{t}):t_{\underline{H},\underline{t}} \to A^1(k(\underline{t}),\ \tilde{\mathfrak{g}}(\underline{t});\tilde{\mathfrak{g}}(\underline{t}))$$

which is therefore surjective. QED.

Compare the above result with condition (V) of §3 [La-Pf]. Applying (3.10) of [La-Pf], we now obtain the following result:

Proposition (2.4). Let $\mathfrak{g}_0 \in \underline{H}$ be a nonsingular point corresponding to a rigid Lie-algebra. Then the component of $\underline{H}$ containing $\mathfrak{g}_0$ is of dimension

$$\dim \mathrm{Der}(\mathfrak{g}) - \dim \mathrm{Der}(\mathfrak{g}_0)$$

Proof. It is clear that there exists a non singular curve $\underline{C}$ mapping into $\underline{H}$, the image of which contains $*$, the point corresponding to $\mathfrak{g}$, and $\mathfrak{g}_0$. Since $\mathfrak{g}_0$ is a nonsingular point of $\underline{H}$ it follows from (3.10) loc.cit. that the dimension of the component containing $\mathfrak{g}_0$ is

$$\dim(\mathrm{Der}(\mathfrak{g})/\mathrm{Der}_{\pi'})$$

where $\mathrm{Der}_{\pi'}$ is the Lie ideal of $\mathrm{Der}(\mathfrak{g})$ consisting of those derivations that lift to the pull back $\mathfrak{g}_C$ of $\tilde{\mathfrak{g}}$ to $\underline{C}$. But, as a C-module $\mathrm{Der}_C(\mathfrak{g}_C) \simeq \mathrm{Der}(\mathfrak{g}_0)\otimes_k C$ since this holds on an open dense subset of $\underline{C}$, and since $\mathrm{Der}_C(\mathfrak{g}_C)$ is flat on $C$. Now $\mathrm{Der}_{\pi'} \simeq \mathrm{Der}_C(\mathfrak{g}_C)\otimes_C k \simeq \mathrm{Der}(\mathfrak{g}_0)$ and the Proposition follows. Q.E.D.

Now consider the case $\mathfrak{g}=k^n$, the abelian Lie-algebra. The family $\mathcal{L}_n$ defined on $\mathrm{Lie}_n$ is miniversal. In fact, by Example 3. $(\mathrm{Lie}_n,\mathcal{L}_n)$ is an algebraization of the formal versal family of $\mathfrak{g}_0=k^n$ and by (2.3) it is formally versal in the sence of Definition (3.6) of [La-Pf]. It is therefore tempting to try to carry over to this situation the treatment of §3 in [La-Pf]. Recall that there are the basic assumptions (V'), which are not satisfied in the case of Lie-algebras. In particular $\underline{\mathrm{Lie}}_n$ is far from non-singular.

In fact the scheme $\underline{\mathrm{Lie}}_n$ is highly complex. There are many components, some of which are non-reduced. There are examples of rigid Lie-algebras $\mathfrak{g}$ that are not semisimple, see f.ex. [Ri], and there are examples of algebras with artinian formal moduli (unfortunately also called rigid in the literature), see [Ra] where one also finds an example of an algebra $\mathfrak{g}$ with vanishing cup products, but which, never the less, has an artinian formal moduli. In this case the higher Massey products must therefore be nonzero, see [La1] (4.3). Notice that, to every Lie-algebra with an artinian formal moduli, there is a component of $\underline{\mathrm{Lie}}_n$ which is non singular if and only if $A^1(k,\mathfrak{g};\mathfrak{g})=0$, i.e. if and only if $\mathfrak{g}$ is rigid, but otherwise generically non reduced.

However, in this case it is clear that $V=\ker g$ is generated by the image of $\mathrm{LieGl}_n(k)=gl_n$. Since $gl_n$ is reductive there exists a stratification $\{\underline{S}_\tau\}$ of $\underline{\mathrm{Lie}}_n$ invariant under $gl_n$, and geometric quotients

$$\underline{M}_\tau = \underline{S}_\tau / gl_n$$

Moreover if $t \in \underline{S}_\tau$ is a closed point then necessarily $\dim \underline{M}_\tau < \dim \underline{H}_0^\wedge$ where $\underline{H}_0^\wedge$ is the prorepresenting substratum of the formal moduli of $\mathfrak{g} = \tilde{\mathfrak{g}}(\underline{t})$. Recall that $H_0^\wedge$ has a tangent space, contained in that of $\underline{H}^\wedge$, which is

$$H^0(\mathrm{Der}(\mathfrak{g}), A^1(k,\mathfrak{g};\mathfrak{g})) \subseteq A^1(k,\mathfrak{g};\mathfrak{g}) .$$

As an immediate result we therefore obtain,

<u>Proposition (2.5)</u>. The component of $\underline{M}_\tau$ containing $\mathfrak{g}$ has dimension less or equal to

$$\dim_k H^0(L(\mathfrak{g}), A^1(k,\mathfrak{g};\mathfrak{g})) .$$

<u>Proof</u>. We only have to recall that $L(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})/\mathrm{Der}_1$, and that $\mathrm{Der}(\mathfrak{g})$ acts on $A^1$ via $L(\mathfrak{g})$. QED.

## <u>§3. Local moduli for isolated hypersurface singularities, and the imbedding problem</u>.

The purpose of this § is the proof in a special case of the following conjecture,

<u>Conjecture (3.1) (The Imbedding Theorem)</u>. Let $f \in k[\underline{x}]$ be an isolated hypersurface singularity, and let $\underline{H}_0$ be the prorepresenting substratum of $\underline{H}$. Denote by $\mathcal{L}$ the flat family of $H_0$-Lie-algebras

$$V \underset{H}{\otimes} H_0$$

where $V$ is the kernel of the Kodaira-Spencer map of the versal family $F$ of $f$. Then for every closed point $\underline{t}\in\underline{H}_0$, the Kodaira-Spencer morphism of the family $\mathcal{L}$ induces a commutative diagram

$$\begin{array}{ccc}
\mathrm{Der}_k(H_0) & \xrightarrow{\lambda} & A^1(H_0,\mathcal{L};\mathcal{L}) \\
\downarrow & & \downarrow \\
t_{H_0,t} & \xrightarrow[\lambda(\underline{t})]{} & A^1(k,L(F(\underline{t}));L(F(\underline{t})))
\end{array} ,$$

and, except for some special cases of low dimensional $\underline{H}_0$'s, there is an open dense subset $U \subseteq \underline{H}_0$ such that $\lambda(\underline{t})$ is injective for all $\underline{t}\in U$.

There are exceptional cases. In fact $\lambda(\underline{t})$ is constant for $f=x^4+y^4$ and for $f=x^3+y^6$. However in both cases $\dim \underline{H}_0=1$.

At the moment we are able to prove the following.

Theorem (3.2): Let $f=x^k+y^\ell$ be a quasi-homogenous plane curve singularity. Suppose the dimension of the prorepresenting substratum $\underline{H}_0 \subseteq \underline{H}$ is $>2$. Then there is an open dense subset $U$ of $\underline{H}_0$ such that whenever $\underline{t}$ is a closed point in $U$ the map $\lambda(\underline{t})$ is injective.

Proof: Let $I=\{(\alpha_1,\alpha_2)\in Z_t^2 \mid 0\leq\alpha_1\leq k-2, 0\leq\alpha_2\leq \ell-2\}$, $I_0=\{\underline{\alpha}\in I \mid |\underline{\alpha}|=1\}$, where $|(\alpha_1,\alpha_2)| = \alpha_1/k + \alpha_2/\ell$. Then $H=k[t_{\underline{\alpha}}]_{\underline{\alpha}\in I}$, $H_0=k[t_{\underline{\alpha}}]_{\underline{\alpha}\in I_0}$

and $F_0 = f + \sum_{\underline{\alpha}\in I_0} t_{\underline{\alpha}} x^{\underline{\alpha}}$ is the miniversal family $F$ restricted to $\underline{H}_0$.

Since the fibers of $\mathcal{L} = V \otimes_H H_0$ are $L(F_0(\underline{t}))$ - see [La-Pf] - it follows from [§4,loc.cit.] that $\mathcal{L}$ is the Lie-algebra given by

$$\operatorname{Ker} \quad H_0[[x,y]] \Big/ \left(\frac{\partial F_0}{\partial x}, \frac{\partial F_0}{\partial y}\right) \xrightarrow{F_0 \cdot} H_0[[x,y]] \Big/ \left(\frac{\partial F_0}{\partial x}, \frac{\partial F_0}{\partial y}\right)$$

where $F_0\cdot$ is multiplication by $F_0$. Since $F_0$ is quasihomogenous we find $\mathcal{L} \simeq H_0[[x,y]] \Big/ \left(\frac{\partial F_0}{\partial x}, \frac{\partial F_0}{\partial y}\right)$. If $\underline{x}^{\underline{\alpha}}, \underline{x}^{\underline{\beta}} \in \mathcal{L}$, then

$$[\underline{x}^{\underline{\alpha}}, \underline{x}^{\underline{\beta}}] = (|\underline{\beta}| - |\underline{\alpha}|) \cdot \underline{x}^{\underline{\alpha}+\underline{\beta}}$$

Now there is an open neighbourhood $U_0$ of $\underline{0}$ in $\underline{H}_0$ such that

$$B = \{\underline{x}^{\underline{\alpha}} \mid \underline{\alpha} \in \underline{I}\}$$

is a basis for $\mathcal{L}$ locally on $U_0$, see [§4 loc.cit.]. Then, for $\underline{\alpha}, \underline{\beta} \in I$ we have

$$[\underline{x}^{\underline{\alpha}}, \underline{x}^{\underline{\beta}}] = \sum_{\underline{\gamma}\in I} c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} \underline{x}^{\underline{\gamma}}$$

where $c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} \in H_0$. Notice that, since $\mathcal{L}$ is a graded $H_0$-module, $c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} = 0$ unless $|\underline{\gamma}| = |\underline{\alpha}| + |\underline{\beta}|$. In particular, if $\underline{\alpha}+\underline{\beta} \in I$, then

$$c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} = \begin{cases} (|\underline{\beta}| - |\underline{\alpha}|) & \underline{\gamma} = \underline{\alpha}+\underline{\beta} \\ 0 & \text{otherwise} \end{cases}$$

For $\underline{t} \in U_0$, $m$ and $m_0 \in \mathbb{Q}$, define $L_m$ to be the subspace of $L(F_0(\underline{t}))$ generated as a vector space by $\{d_{\underline{\alpha}} \mid \underline{\alpha} \in I, |\underline{\alpha}| = m\}$, and put,

$$L_{>m_0} = \bigoplus_{m>m_0} L_m . \quad L_{>m_0} \text{ is an ideal in } L(F_0(\underline{t})).$$

We shall consider the $H_0$-Lie-algebra $\mathcal{C}^1L(x^k+y^\ell)$, the fibers of which are $\mathcal{C}^1L(F_0(\underline{t}))$. We may assume $k<\ell$, then $\mathcal{C}^1L(F_0(\underline{t}))=L_{>1/\ell}$. From now on, denote $\mathcal{C}^1L(F_0(\underline{t}))$ by $\mathfrak{g}(\underline{t})$.

Notice that $\mathfrak{g}(\underline{t})$ is generated as a vector space by

$$B = \{d_{\underline{\alpha}} \mid \underline{\alpha}\in I^*=I\setminus\{\underline{0}\}\}$$

For $\underline{t}\in U_0$ we may express the Kodaira-Spencer map at $\underline{t}$ as

$$\lambda(\underline{t}) : \mathrm{Der}_k(H_0,k(\underline{t})) \to A^1(k,\mathfrak{g}(\underline{t}),\mathfrak{g}(\underline{t}))$$

An element in $A^1(k,\mathfrak{g}(\underline{t}),\mathfrak{g}(\underline{t}))$ is represented by a homomorphism of F-modules $\eta:\mathfrak{J}\to\mathfrak{g}(\underline{t})$, where $F$ is the free k-Lie-algebra generated by $\{\Delta_{\underline{\alpha}} \mid \underline{\alpha}\in I^*=I\setminus\{\underline{0}\}\}$, $\pi:F\to\mathfrak{g}(\underline{t})$ is given by $\pi(\Delta_{\underline{\alpha}})=d_{\underline{\alpha}}$ and $\mathfrak{J}$ is the ideal of $F$ generated by .

$$\{f_{\underline{\alpha\beta}}=[\Delta_{\underline{\alpha}},\Delta_{\underline{\beta}}]-\sum_{\underline{\gamma}\in I} c^{\underline{\gamma}}\Delta_{\underline{\gamma}} \mid \underline{\alpha},\underline{\beta}\in I^*\}$$

Then, if $\delta\in\mathrm{Der}_k(H_0,k(\underline{t}))$ its image in $A^1(k,\mathfrak{g}(\underline{t}),\mathfrak{g}(\underline{t}))$ is represented by the F-homomorphism given by

$$f_{\underline{\alpha\beta}} \to \lambda_{\underline{\alpha\beta}}=\sum_{\underline{\gamma}\in I}\delta(c^{\underline{\gamma}}_{\underline{\alpha\beta}})d_{\underline{\gamma}}\,, \qquad \underline{\alpha},\underline{\beta}\in I^*$$

Furthermore, $\lambda(\underline{t})(\delta)$ is zero iff

$$(3.3) \qquad \lambda_{\underline{\alpha\beta}} = D(f_{\underline{\alpha\beta}}) \ \forall\ \underline{\alpha},\underline{\beta}\in I^*$$

for some derivation $D:F\to\mathfrak{g}(\underline{t})$. Of course, any derivation $D$ is determined by the images $D(\Delta_{\underline{\alpha}})=v_{\underline{\alpha}}$ in $\mathfrak{g}(\underline{t})$, $\underline{\alpha}\in I^*$. Suppose $D$ satisfies (3.3). Substituting $[\Delta_{\underline{\alpha}},\Delta_{\underline{\beta}}]-\sum_{\underline{\gamma}\in I}c^{\underline{\gamma}}_{\underline{\alpha\beta}}\Delta_{\underline{\gamma}}$ for $f_{\underline{\alpha\beta}}$ we find that, for each $\underline{\alpha},\underline{\beta}$:

$$(3.4) \qquad [v_{\underline{\alpha}},d_{\underline{\beta}}] + [d_{\underline{\alpha}},v_{\underline{\beta}}] - \sum_{\underline{\gamma}\in I}c^{\underline{\gamma}}_{\underline{\alpha\beta}}v_{\underline{\gamma}} = \lambda_{\underline{\alpha\beta}}$$

where $v_{\underline{\alpha}}=D(\Delta_{\underline{\alpha}})$. Now a closer examination shows:

<u>Lemma 1</u>: If $D$ is any derivation satisfying (3.3), then

$$D(\Delta_{\underline{\alpha}}) \in L_{>|\underline{\alpha}|} \quad \forall\ \underline{\alpha} \in I^*.$$

<u>Proof</u>: If $\underline{\alpha}+\underline{\beta} \in I^*$, then $f_{\underline{\alpha\beta}} = [\Delta_{\underline{\alpha}}, \Delta_{\underline{\beta}}] - (|\underline{\beta}|-|\underline{\alpha}|)\Delta_{\underline{\alpha}+\underline{\beta}}$, $\lambda_{\underline{\alpha\beta}}=0$. In this case, (3.4) reads

$$(3.5) \qquad [v_{\underline{\alpha}}, d_{\underline{\beta}}] + [d_{\underline{\alpha}}, v_{\underline{\beta}}] - (|\underline{\beta}|-|\underline{\alpha}|)v_{\underline{\alpha}+\underline{\beta}} = 0$$

Since $[L_{>|\underline{\alpha}|}, L_{>|\underline{\beta}|}] \subseteq L_{>|\underline{\alpha}|+|\underline{\beta}|}$ we only need to prove that $v_{\underline{\alpha}} \in L_{>|\underline{\alpha}|}$ $\forall\ \underline{\alpha} \in J$, where $\{d_{\underline{\alpha}} | \underline{\alpha} \in J\}$ generate $\mathfrak{g}(\underline{t})$ as a Lie-algebra. Using the equations (3.5), this is straightforward computation, which we shall leave to the reader. QED.

Recall that, on $U_0$, $c^{\underline{\gamma}}_{\underline{\alpha\beta}}=0$ whenever $|\underline{\gamma}| \neq |\underline{\alpha}|+|\underline{\beta}|$. Hence

$$\lambda_{\underline{\alpha\beta}} \in L_{|\underline{\alpha}|+|\underline{\beta}|} \quad \forall\ \underline{\alpha}, \underline{\beta} \in I^*.$$

Thus, we obtain the,

<u>Corollary</u>: Suppose $\lambda_{\underline{\alpha\beta}} = D(f_{\underline{\alpha\beta}})$ for some derivation $D$. Then there is a derivation $D'$ such that

1) $D'(f_{\underline{\alpha\beta}}) = \lambda_{\underline{\alpha\beta}}$ $\forall\ \underline{\alpha}, \underline{\beta} \in I^*$
2) $D'(\Delta_{\underline{\alpha}}) \in L_{|\underline{\alpha}|}$ $\forall\ \underline{\alpha} \in I$

<u>Proof</u>: if $D(\Delta_{\underline{\alpha}}) = v'_{\underline{\alpha}} + w_{\underline{\alpha}}$, $v'_{\underline{\alpha}} \in L_{|\underline{\alpha}|}$, $w_{\underline{\alpha}} \in L_{>|\underline{\alpha}|}$, define $D'$ by $D'(\Delta_{\underline{\alpha}}) = v'_{\underline{\alpha}}$. Q.E.D.

Our aim is to prove that if $\lambda(\underline{t})(\delta)$ is the restriction of some derivation $D: F \to \mathfrak{g}(\underline{t})$, then $\delta=0$. By the corollary, we may assume $D(\Delta_{\underline{\alpha}}) = v_{\underline{\alpha}} \in L_{|\underline{\alpha}|}$ $\forall\ \underline{\alpha} \in I^*$. We may write $f=x^{an}+y^{bn}$, where $a<b$, $a$ and $b$ are relatively prime. For any $\underline{\alpha} \in Z^2$, let $\underline{\alpha}_+ = \underline{\alpha}+(a,-b)$, $\underline{\alpha}_- = \underline{\alpha}-(a,-b)$.

From now on, let $D$ be any derivation satisfying (3.3), $D(\Delta_{\underline{\alpha}}) \in L_{|\underline{\alpha}|}$. Let $v_{\underline{\alpha}} = D(\Delta_{\underline{\alpha}})$. Then, in particular, we may express the terms $v_{(1,0)}$ and $v_{(0,1)}$ as linear combinations of the generators $d_{\underline{\alpha}}$ of weight $\frac{1}{k}$, $\frac{1}{\ell}$, respectively.
Now for each weight $\frac{1}{k}$, $\frac{1}{\ell}$ there are only one or two elements $\underline{\alpha} = (\alpha_1, \alpha_2)$ in $I$, the number depending on $k$, $\ell$. In any case we may write

$$(3.6) \qquad \begin{aligned} v_{(1,0)} &= a_{22} d_{(1,0)} + a_{21} d_{(1,0)_-} \\ v_{(0,1)} &= a_{11} d_{(0,1)} + a_{12} d_{(0,1)_+} \end{aligned}$$

where the $a_{ij}$'s are unknown scalars. Notice that if $b>2$, then $(0,1)_+ \notin Z^2_+$, in which case $a_{12}$ is necessarily 0. If, in addition, $a>2$ $(a<b)$, then $a_{12}$ and $a_{21}$ are both 0. Bearing this in mind, we have

Lemma 2: Let $D$ be as prescribed, $a_{ij}$ as in (3.6), $i,j=1,2$.
Let $\underline{\alpha} = (\alpha_1, \alpha_2) \in I^*$, such that

i) $\underline{\alpha}_+ \in I^*$ or $\alpha_2 \cdot a_{12} = 0$

and ii) $\underline{\alpha}_- \in I^*$ or $\alpha_1 \cdot a_{21} = 0$

Then

$$v_{\underline{\alpha}} = \alpha_2 \cdot a_{12} d_{\underline{\alpha}_+} + (\alpha_2 \cdot a_{11} + \alpha_1 \cdot a_{22}) d_{\underline{\alpha}} + \alpha_1 \cdot a_{21} d_{\underline{\alpha}_-}$$

Proof: Once again, it suffices to prove the formula for $\underline{\alpha} \in J$, where $\{d_{\underline{\alpha}} | \underline{\alpha} \in J\}$ generate $\mathfrak{g}(\underline{t})$ as a Lie-algebra. If $\underline{\gamma} = (\gamma_1, \gamma_2)$ satis- fies the conditions i) and ii), one easily sees that either $\underline{\gamma} \in J$, or $\underline{\gamma} = \underline{\alpha} + \underline{\beta}$, where $|\underline{\alpha}| \neq |\underline{\beta}|$, and $\underline{\alpha}, \underline{\beta}$ satisfies i) and ii). In the latter case, by (3.5)

$$v_{\underline{\gamma}} = \frac{1}{|\underline{\beta}| - |\underline{\alpha}|} \cdot ([v_{\underline{\alpha}}, d_{\underline{\beta}}] + [d_{\underline{\alpha}} \cdot v_{\underline{\beta}}])$$

Since $|\underline{\alpha}|, |\underline{\beta}| < |\underline{\gamma}|$ we may argue by induction on $|\underline{\gamma}|$, obtaining

$$v_{\underline{\gamma}} = \frac{1}{|\underline{\beta}|-|\underline{\alpha}|}\cdot([\alpha_2\cdot a_{12}d_{\underline{\alpha}_+}+(\alpha_2\cdot a_{11}+\alpha_1\cdot a_{22})d_{\underline{\alpha}}+\alpha_1\cdot a_{21}d_{\underline{\alpha}_-}, d_{\underline{\beta}}]$$
$$+ [d_{\underline{\alpha}}, \beta_2\cdot a_{12}d_{\underline{\beta}_+}+(\beta_2\cdot a_{11}+\beta_1\cdot a_{22})d_{\underline{\beta}}+\beta_1\cdot a_{21}d_{\underline{\beta}_-}])$$
$$= \gamma_2\cdot d_{\underline{\gamma}_+} + \gamma_2\cdot a_{11}+\gamma_1\cdot a_{22})d_{\underline{\gamma}} + \gamma_1\cdot a_{21}d_{\underline{\gamma}_-}$$

(since $\gamma_i=\alpha_i+\beta_i$, $i=1,2$; $[d_{\underline{\alpha}_+}, d_{\underline{\beta}}]=[d_{\underline{\alpha}}, d_{\underline{\beta}_+}]=d_{\underline{\gamma}_+}$ and $[d_{\underline{\alpha}_-}, d_{\underline{\beta}}]=$ $[d_{\underline{\alpha}}, d_{\underline{\beta}_-}]=d_{\underline{\gamma}_-}$).

Proving the formula for $\underline{\alpha}\in J$ is a rather straightforward matter, though the explicite calculations depend on $k$, $\ell$. A typical example of the calculations involved is given by the case $f(x,y) = x^n+y^n$:

Consider, by (3.5), the equations

$$[v_{(1,0)}, d_{(0,2)}]+[d_{(1,0)}, v_{(0,2)}] = \frac{1}{n}\cdot v_{(1,2)}$$
$$[v_{(0,1)}, d_{(1,1)}]+[d_{(0,1)}, v_{(1,1)}] = \frac{1}{n}\cdot v_{(1,2)}$$

Express the $v_{\underline{\alpha}}$'s in terms of the generators $d_{\underline{\alpha}}$, then compare coefficients. The other equations are similar.
In fact, this step is where the condition $\dim \underline{H}_0 > 2$ is needed, restricting the "smallness" of $k$, $\ell$.

Q.E.D.

For any $\underline{\alpha},\underline{\beta}\in I^*$ and any derivation $D$ we have

(3.7) $$D(f_{\underline{\alpha\beta}}) = [v_{\underline{\alpha}}, d_{\underline{\beta}}] + [d_{\underline{\alpha}}, v_{\underline{\beta}}] - \sum_{\underline{\gamma}\in I^*} c^{\underline{\gamma}}_{\underline{\alpha\beta}} v_{\underline{\gamma}}$$

Choose $\underline{\alpha},\underline{\beta}$ such that the formula of Lemma 2 holds. Then

(3.8) $$[v_{\underline{\alpha}}, d_{\underline{\beta}}] + [d_{\underline{\alpha}}, v_{\underline{\beta}}]$$
$$= \sum_{\underline{\gamma}\in I}(c^{\underline{\gamma}}_{\underline{\alpha\beta}_+}(\alpha_2+\beta_2)a_{12}+c^{\underline{\gamma}}_{\underline{\alpha\beta}}((\alpha_1+\beta_1)a_{22}+(\alpha_2+\beta_2)a_{11})+c^{\underline{\gamma}}_{\underline{\alpha\beta}_-}(\alpha_1+\beta_1)a_{21})d_{\underline{\gamma}}$$

(Notice that $c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}_+} = c^{\underline{\gamma}}_{\underline{\alpha}_+\underline{\beta}}$, $c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}_-} = c^{\underline{\gamma}}_{\underline{\alpha}_-\underline{\beta}}$ $\forall\ \underline{\gamma}\in I$).

Now one easily checks that for any $d_{\underline{\gamma}}\in \mathcal{C}^1\mathfrak{g}(\underline{t})$, there are $\underline{\alpha}_{\underline{\gamma}}, \underline{\beta}_{\underline{\gamma}} \in I^*$ satisfying $|\underline{\alpha}_{\underline{\gamma}}| \neq |\underline{\beta}_{\underline{\gamma}}|$, $\underline{\alpha}_{\underline{\gamma}} + \underline{\beta}_{\underline{\gamma}} = \underline{\gamma}$ and $v_{\underline{\alpha}_{\underline{\gamma}}}, v_{\underline{\beta}_{\underline{\gamma}}}$ as in Lemma 2. Write $\underline{\alpha}_{\underline{\gamma}} = (\alpha_{1\underline{\gamma}}, \alpha_{2\underline{\gamma}})$, $\beta_{\underline{\gamma}} = (\beta_{1\underline{\gamma}}, \beta_{2\underline{\gamma}})$, then by (3.5)

$$\text{(3.9)}\qquad v_{\underline{\gamma}} = [(\alpha_{2\underline{\gamma}}+\beta_{2\underline{\gamma}})a_{11} + (\alpha_{1\underline{\gamma}}+\beta_{1\underline{\gamma}})a_{22}]d_{\underline{\gamma}}$$
$$+ \frac{1}{|\underline{\beta}|-|\underline{\alpha}|}\sum_{\underline{\delta}\in I}[c^{\underline{\delta}}_{\underline{\alpha}_{\underline{\gamma}}\underline{\beta}_{\underline{\gamma}_+}}(\alpha_{2\underline{\gamma}}+\beta_{2\underline{\gamma}})a_{12} + c^{\underline{\delta}}_{\underline{\alpha}_{\underline{\gamma}}\underline{\beta}_{\underline{\gamma}_-}}(\alpha_{1\underline{\gamma}}+b_{1\underline{\gamma}})a_{21}]d_{\underline{\delta}}$$

Combining (3.8) and (3.9) we obtain

$$\text{(3.10)}\qquad D(f_{\underline{\alpha}\underline{\beta}}) =$$

$$\sum_{\underline{\gamma}\in I}\{[\sum_{\underline{\delta}\in I}\frac{c^{\underline{\delta}}_{\underline{\alpha}\underline{\beta}}}{|\underline{\beta}|-|\underline{\alpha}|}\cdot c^{\underline{\gamma}}_{\underline{\alpha}_{\underline{\delta}}\underline{\beta}_{\underline{\delta}_+}}\cdot(\alpha_{2\underline{\delta}}+\beta_{2\underline{\delta}}) + c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}_+}\cdot(\alpha_2+\beta_2)]a_{12}$$
$$+ c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}\cdot[(\alpha_2+\beta_2-\alpha_{2\underline{\gamma}}-\beta_{2\underline{\gamma}})a_{11} + (\alpha_1+\beta_1-\alpha_{1\underline{\gamma}}-\beta_{1\underline{\gamma}})a_{22})]$$
$$+ [\sum_{\underline{\delta}\in I}\frac{c^{\underline{\delta}}_{\underline{\alpha}\underline{\beta}}}{|\underline{\beta}|-|\underline{\alpha}|}\cdot c^{\underline{\gamma}}_{\underline{\alpha}_{\underline{\delta}}\underline{\beta}_{\underline{\delta}_-}}\cdot(\alpha_{1\underline{\delta}}+\beta_{1\underline{\delta}}) + c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}_-}\cdot(\alpha_1+\beta_1)]a_{21}\}d_{\underline{\gamma}}$$

Recall that $c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}} = 0$ unless $|\underline{\alpha}|+|\underline{\beta}| = |\underline{\gamma}|$. Hence $|\underline{\alpha}_{\underline{\gamma}}+\underline{\beta}_{\underline{\gamma}}| = |\underline{\gamma}| = |\underline{\alpha}|+|\underline{\beta}|$ for each term in the sum, and we calculate

$$(\alpha_1+\beta_1-\alpha_{1\underline{\gamma}}-\beta_{1\underline{\gamma}}) = -\frac{k}{\ell}(\alpha_2+b_2-\alpha_{2\underline{\gamma}}-\beta_{2\underline{\gamma}})$$

Therefore, the second term of the right hand side of (3.10) is

$$\text{(3.10.1)}\qquad c^{\underline{\gamma}}_{\underline{\alpha}\underline{\beta}}\cdot(\alpha_2+\beta_2-\alpha_{2\underline{\gamma}}-\beta_{2\underline{\gamma}})\cdot(a_{11} - \frac{k}{\ell}a_{22})$$

We started by assuming that $D$ is a derivation such that $D(f_{\underline{\alpha}\underline{\beta}}) - \lambda_{\underline{\alpha}\underline{\beta}} = 0$ $\forall\ \underline{\alpha},\underline{\beta}\in I^*$. If $\delta = \sum_{\underline{z}\in I_0} u_{\underline{z}}\frac{\partial}{\partial t_{\underline{z}}} \in \mathrm{Der}_k(H_0, k(\underline{t}))$, then

$\lambda(\underline{t})(\delta)$ is given by

$$(3.11) \qquad \lambda_{\underline{\alpha\beta}} = \sum_{\underline{\gamma}\in I} \Big( \sum_{\underline{z}\in I_0} \frac{\partial}{\partial t_{\underline{z}}} (c^{\gamma}_{\underline{\alpha\beta}}) \cdot u_{\underline{z}}$$

Comparing coefficients for each $d\underline{\gamma}$ we find, by (3.10), (3.10.1) and (3.11) that the equations (3.3) turn out to be nothing but a homogenous system of linear equations in $a_{12}, a_{21}, (a_{11} - \frac{k}{\ell}a_{22})$ and $\{u_{\underline{z}} | \underline{z}\in I_0\}$.

We shall choose suitable equations and compute the determinant of the resulting square matrix

$$M = \begin{bmatrix} \phi_{11} & \cdots\cdots & \phi_{1s} \\ \vdots & & \vdots \\ \phi_{s1} & \cdots\cdots & \phi_{ss} \end{bmatrix}$$

In fact, we only need compute this determinant modulo $(t_{\underline{\alpha}})^2$ and show that the first-order term is non-zero. Choosing equations carefully, we find $\phi_{j1}(\underline{0})=0$, $j=1,\ldots,s$, so that, modulo $(t_{\underline{\alpha}})^2$, $\det(M)$ equals

$$\det \begin{bmatrix} \bar{\phi}_{11} & \phi_{12}(\underline{0}) \cdots & \phi_{1s}(\underline{0}) \\ \vdots & & \vdots \\ \bar{\phi}_{s1} & \phi_{s2}(\underline{0}) \cdots & \phi_{ss}(\underline{0}) \end{bmatrix}$$

where $\bar{\phi}_{j1}$ is the first-order term of $\phi_{j1}$.

To compute the latter matrix, observe that by (3.10), (3.10.1) and (3.11), the coefficients $\phi_{ij}$ are simple polynomials in the $c^{\gamma}_{\underline{\alpha\beta}}$'s and their partial derivatives. Finally, notice that for $\underline{t} = \underline{0}$, $L(F_0(\underline{0})) = L(f)$. Since, in $L(f)$

$$[d_{\underline{\alpha}}, d_{\underline{\beta}}] = 0 \qquad \text{if} \quad \underline{\alpha}+\underline{\beta} \notin I$$

we conclude that the structural constants $c^{\underline{\delta}}_{\underline{\alpha\beta}}$ in $\mathcal{L}$ are 0 modulo $(t_{\underline{\alpha}})$ whenever $\underline{\alpha}+\underline{\beta} \notin I$.

Taking advantage of this fact, we may use the fundamental relations

$$\frac{\partial F_0}{\partial x} = 0 = \frac{\partial F_0}{\partial y} \quad \text{in} \quad H_0[[x,y]]/(\frac{\partial F_0}{\partial x}, \frac{\partial F_0}{\partial y}) \simeq \mathcal{L},$$

to read off the first-order term of any $c^{\underline{\gamma}}_{\underline{\alpha\beta}}$.

Since $\dim \underline{H}_0 > 2$ there are 3 cases to consider:

1) $f = x^n + y^n \quad , \quad n>5$

2) $f = x^n + y^{an} \quad , \quad n>4, \ a>2$

3) $f = x^{an} + y^{bn} \quad , \quad n>3, \ 2<a<b, \ (a,b)=1$

In the second case, the expression (3.10) is simplified by the fact that the variable $a_{12}$ is necessarily 0, whereas in case 3), $a_{12} = a_{21} = 0$ (recall the discussion preceding Lemma 2). Hence, the explicite equations (even the number of variables involved) differ slightly in each of the three cases.

For this reason, it turns out more convenient to consider each case separately.

<u>Case 1): $f(x,y)=x^n+y^n$, $n>5$.</u>

In this case, $H_0=k[t_2,\dots,t_{n-2}]$, $\dim H_0=n-3$

$$F_0 = x^n + y^n + t_2x^{n-2}y^2 + t_3x^{n-3}y^3 + \dots + t_{n-2}x^2y^{n-2}$$

$$\text{i)} \quad \frac{1}{n}\cdot\frac{\partial F_0}{\partial x} = x^{n-1} + \frac{n-2}{n}t_2x^{n-3}y^2 + \frac{n-3}{n}t_3x^{n-4}y^3 + \dots + \frac{2}{n}t_{n-2}xy^{n-2}$$

(3.12)

$$\text{ii)} \quad \frac{1}{n}\cdot\frac{\partial F_0}{\partial y} = y^{n-1} + \frac{2}{n}t_2x^{n-2}y + \frac{3}{n}t_3x^{n-3}y^2 + \dots + \frac{n-2}{n}t_{n-2}x^2y^{n-3}$$

On $U_0$, $\mathcal{B}=\{x^{\alpha_1}y^{\alpha_2}\,|\,0<\alpha_i<n-2\}$ is a basis for $H_0[[x,y]]/(\frac{\partial F_0}{\partial x},\frac{\partial F_0}{\partial y})\simeq\mathcal{L}$.

In B, the only monomials of weight $\frac{2n-5}{n}$ are $x^{n-2}y^{n-3}$, $x^{n-3}y^{n-2}$. Any other monomial of weight $\frac{2n-5}{n}$ may be expressed as an $H_0$-linear combination of $x^{n-2}y^{n-3}$, $x^{n-3}y^{n-2}$:

$$x^{\ell}y^{2n-5-\ell}=\Omega_{\ell}\cdot x^{n-2}y^{n-3}+\chi_{\ell}\cdot x^{n-3}y^{n-2}\ ,\qquad \ell=0,1,\dots,2n-5$$

For $\ell\neq n-2,n-3$ $\Omega_\ell$ and $\chi_\ell\in(t_2,\dots,t_{n-2})$. Using (3.12) i) and ii) we find that, modulo $(t_2,\dots,t_{n-2})^2$

$$\text{(3.13)}\qquad \begin{aligned}
&\Omega_{(2n-k)}=-\frac{k-2}{n}t_{n+2-k}\ ,\ \chi_{2n-k}=-\frac{k-3}{n}t_{n+3-k}\ ,\ k=5,\dots,n\\
&\Omega_{n-1}=0\ ,\ \chi_{n-1}=-\frac{n-2}{n}t_2\\
&\Omega_{n-4}=-\frac{n-2}{n}t_{n-2}\ ,\ \chi_{n-4}=0\\
&\Omega_k=-\frac{k+2}{n}t_{k+2}\ ,\ \chi_k=-\frac{k+3}{n}t_{k+3}\ ,\qquad k=0,\dots,n-5
\end{aligned}$$

Finally, of course $\Omega_{n-2}=\chi_{n-3}=1$,
$\Omega_{n-3}=\chi_{n-2}=0$.

Using (3.13) we may also find the constant term of each partial derivative $\frac{\partial\Omega_\ell}{\partial t_i}$, $\frac{\partial\chi_\ell}{\partial t_i}$.

Next, notice that, on $U_0$,

$$[x^{n-3-k}y^k,x^{n-2-\ell}\cdot y^\ell]=\frac{1}{n}\Omega_{2n-5-k-\ell}x^{n-2}y^{n-3}+\frac{1}{n}\chi_{2n-5-k-\ell}x^{n-3}y^{n-2}$$

Hence, in the F-ideal $\mathcal{J}$,

$$\text{(3.14)}\qquad \begin{aligned}f_{(n-3-k,k)(n-2-\ell,\ell)}&=[\Delta_{(n-3-k,k)},\Delta_{(n-2-\ell,\ell)}]\\
&\quad-\frac{1}{n}\Omega_{2n-5-k-\ell}\Delta_{(n-2,n-3)}-\frac{1}{n}\chi_{2n-5-k-\ell}\Delta_{(n-3,n-2)}\end{aligned}$$

In $\mathfrak{g}(\underline{t})$, write $A=d_{(n-2,n-3)}$, $B=d_{(n-3,n-2)}$

$$v_A = v_{(n-2,n-3)} \;,\quad v_B = v_{(n-3,n-2)}$$

In this case, $\mathfrak{g}(\underline{t})$ is generated as a Lie-algebra by $d_{(1,0)}$, $d_{(0,1)}$, $d_{(2,0)}$, $d_{(1,1)}$, $d_{(0,2)}$. Then (3.6) reads

$$\begin{aligned} v_{(1,0)} &= a_{22}d_{(1,0)} + a_{21}d_{(0,1)} \\ v_{(0,1)} &= a_{11}d_{(0,1)} + a_{12}d_{(1,0)} \end{aligned}$$

where each $a_{ij}$ is an undetermined scalar.

We shall examine the system of linear equations arising from the equations

$$D(f_{\underline{\alpha\beta}}) - \lambda_{\underline{\alpha\beta}} = 0$$

where $|\underline{\alpha}| = \frac{n-3}{n}$, $|\underline{\beta}| = \frac{n-2}{n}$.

By (3.10) and (3.11), the structural constants in $\mathcal{L}$ involved in these equations are just the functions in $H_0$ which we have denoted by $\Omega_\ell, \chi_\ell$.

Now if $\underline{\alpha},\underline{\beta}\in I$, $|\underline{\alpha}| = \frac{n-3}{n}$, $|\underline{\beta}| = \frac{n-2}{n}$ then

$$\underline{\alpha} = (n-3-k,k),\; \underline{\beta} = (n-2-\ell,\ell) \qquad \begin{array}{l} 0<k<n-3 \\ 0<\ell<n-2 \end{array}$$

By (3.14),

$$\begin{aligned} D(f_{(n-3-k,k)(n-2-\ell,\ell)}) = {} & [d_{(n-3-k,k)}, v_{(n-2-\ell,\ell)}] \\ & + [v_{(n-3-k,k)}, d_{(n-2-\ell,\ell)}] \\ & - \frac{1}{n}\Omega_{(2n-5-k-\ell)}v_A - \frac{1}{n}\chi_{(2n-5-k-\ell)}v_B \end{aligned}$$

Obviously, $v_{(n-3-k,k)}$ and $v_{(n-2-\ell,\ell)}$ are given in terms of the $a_{ij}$'s by Lemma 2. Although the formula of Lemma 2 does not apply for $v_A, v_B$, we may still use (3.5) to get the following expressions:

$$
(3.15)\quad
\begin{aligned}
v_A &= [(n-3)(a_{11}+\Omega_{n-1}a_{12})+(n-2)a_{22}]\cdot A \\
&\quad + [(n-3)\chi_{n-1}a_{12}+(n-2)a_{21}]\cdot B \\
v_B &= [(n-3)\Omega_{n-4}a_{21}+(n-2)a_{12}]\cdot A \\
&\quad + [(n-3)(a_{22}+\chi_{n-4}a_{21})+(n-2)a_{11}]\cdot B
\end{aligned}
$$

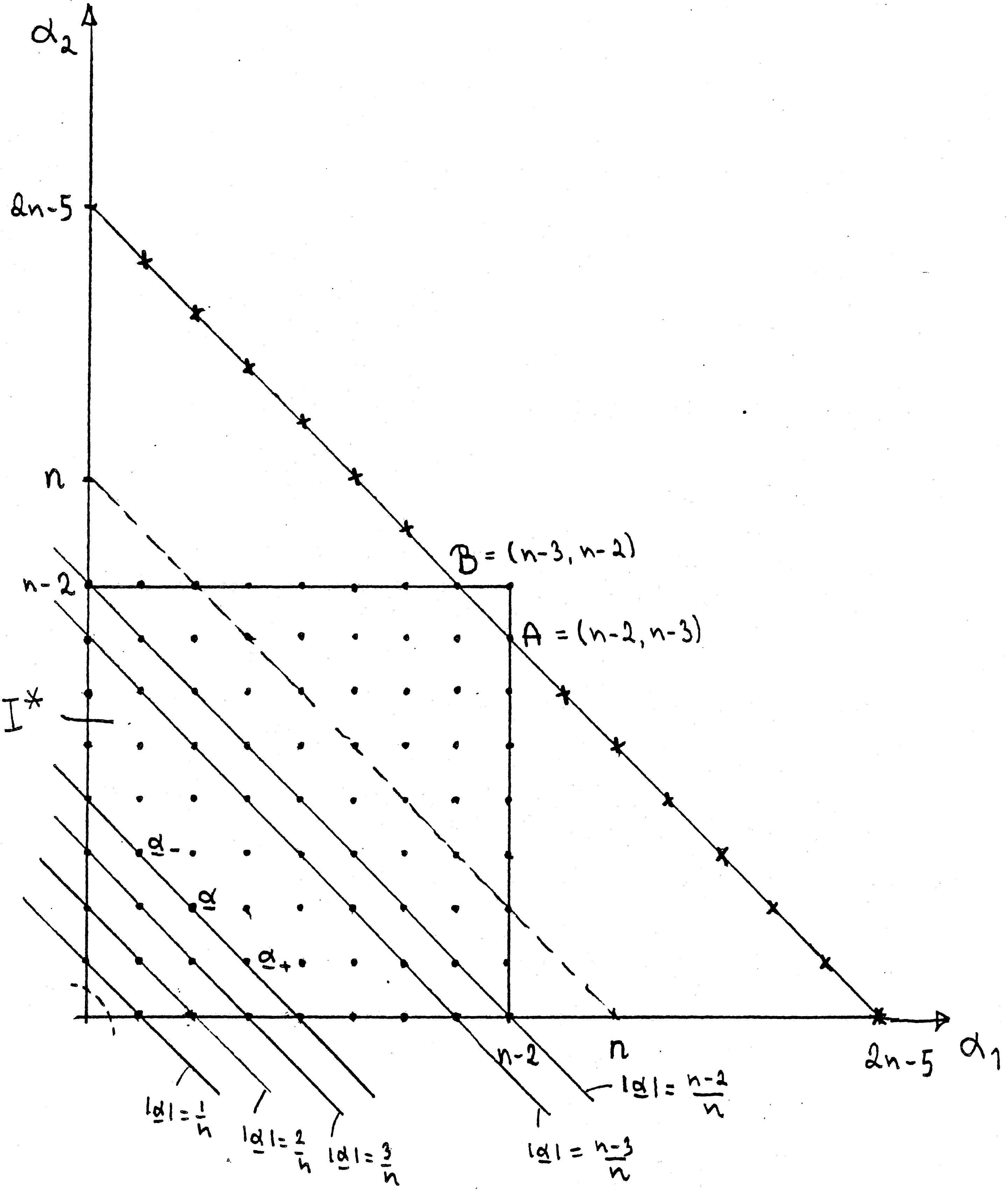
Substituting for $v_{(n-3-k,k)}, v_{(n-2-\ell,\ell)}, v_A$ and $v_B$, by another simple calculation we finally obtain

$$
(3.16)\quad
\begin{aligned}
&n\cdot D(f_{(n-3-k,k)(n-2-\ell,\ell)}) = \\
&\{(k+\ell+3-n)\cdot\Omega_{2n-5-k-\ell}\cdot(a_{11}-a_{22}) \\
&+[(k+\ell)\cdot\Omega_{2n-4-k-\ell}+(3-n)\cdot\Omega_{n-1}\cdot\Omega_{2n-5-k-\ell}+(2-n)\chi_{2n-5-k-\ell}]\cdot a_{12} \\
&+[(2n-5-k-\ell)\cdot\Omega_{2n-6-k-\ell}+(3-n)\cdot\Omega_{n-4}\cdot\chi_{2n-5-k-\ell}]\cdot a_{21}\}A \\
&+\{(k+\ell+2-n)\cdot\chi_{2n-5-k-\ell}\cdot(a_{11}-a_{22}) \\
&+[(k+\ell)\cdot\chi_{2n-4-k-\ell}+(3-n)\cdot\chi_{n-1}\cdot\Omega_{2n-5-k-\ell}]\cdot a_{12} \\
&+[(2n-5-k-\ell)\chi_{2n-6-k-\ell}+(2-n)\Omega_{n-4}+(3-n)\chi_{n-4}\cdot\chi_{2n-5-k-\ell}]a_{21}\}\cdot B
\end{aligned}
$$

where $A=d_{(n-2,n-3)}$, $B=d_{(n-3,n-2)}$.

At this stage, a simple sketch of $I \subseteq Z_+^2$ might be helpful to illustrate what we are really doing:

Fig. 1 $\quad f(x,y) = x^n + y^n \quad (n=10)$



In the diagram, each generator $d_{\underline{\alpha}}$ of weight $w$ is represented by the integer point $\underline{\alpha}$ in $I^*$ lying on the line

$$|\underline{\alpha}| = \frac{\alpha_1}{n} + \frac{\alpha_2}{n} = w$$

Furthermore, for any $\underline{\alpha},\underline{\beta} \in I^*$, $|\underline{\alpha}| \neq |\underline{\beta}|$, the Lie product $[d_{\underline{\alpha}}, d_{\underline{\beta}}]$ is represented by the point $\underline{\alpha}+\underline{\beta}$ of weight $|\underline{\alpha}|+|\underline{\beta}|$.

(The example shown is $f(x,y) = x^{10}+y^{10}$)

Finally, we notice that if $\delta = \sum_{i=2}^{n-2} u_i \cdot \frac{\partial}{\partial t_i} \in \mathrm{Der}_k(H_0, k(\underline{t}))$ then

$$n\cdot\lambda_{(n-3-k,k)(n-2-\ell,\ell)} = \left(\sum_{i=2}^{n-2} \frac{\partial\Omega_{2n-5-k-\ell}}{\partial t_i}\cdot u_i\right)\cdot A + \left(\sum_{i=2}^{n-2} \frac{\partial\chi_{2n-5-k-\ell}}{\partial t_i}\cdot u_i\right)\cdot B \tag{3.17}$$

Consider the system of linear equations in $(a_{11}-a_{22})$, $a_{12}$, $a_{21}$, $u_2, u_3, \ldots u_{n-2}$ arising by comparing coefficients in the equations

$n\cdot(D(f_{(n-4,1)(0,n-2)}) - \lambda_{(n-4,1)(0,n-2)}) = 0$ A-component

$n\cdot(D(f_{(0,n-3)(n-4,2)}) - \lambda_{(0,n-3)(n-4,2)}) = 0$ B-component

$n\cdot(D(f_{(n-3,0)(n-2-\ell,\ell)}) - \lambda_{(n-3,0)(n-2-\ell,\ell)}) = 0$ A-component, $\ell=0,\ldots,n-4$

$n\cdot(D(f_{(n-3,0)(n-2,0)}) - \lambda_{(n-3,0)(n-2,0)}) = 0$ B-component

If we consider the $n\times n$ coefficient matrix associated to the resulting system of equations, then each entry in the first column is 0 modulo $(t_2,\ldots,t_{n-2})$. Hence, if the matrix is

$$M = \begin{bmatrix} \phi_{11} & \cdots & \phi_{1n} \\ \vdots & & \vdots \\ \phi_{n1} & \cdots & \phi_{nn} \end{bmatrix}$$

then the first order term of $\det(M)$ is the determinant of

$$\bar{M} = \begin{bmatrix} \bar{\phi}_{11} & \phi_{12}(\underline{0})\cdots\phi_{1n}(\underline{0}) \\ \vdots & \vdots \\ \bar{\phi}_{n1} & \phi_{n2}(\underline{0})\cdots\phi_{nn}(\underline{0}) \end{bmatrix}$$ where $\bar{\phi}_{ij}=\phi_{ij}$ modulo $(t_2,\ldots,t_{n-2})^2$

Computing $\bar{M}$ is easy, using (3.13), (3.16) and (3.17). We find

$$\bar{M} = \begin{bmatrix} \frac{2(2-n)}{n}t_{n-2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{(n-2)}{n} \\ 0 & (n-1) & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & (n-1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{2(n-2)}{n}t_2 & 0 & 0 & \frac{(n-2)}{n} & 0 & \cdots & 0 & 0 & 0 \\ \frac{3(n-3)}{n}t_3 & 0 & 0 & 0 & \frac{(n-3)}{n} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & & \\ \frac{(n-3)3}{n}t_{n-3} & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{3}{n} & 0 \\ \frac{(n-2)2}{n}t_{n-2} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{2}{n} \end{bmatrix}$$

and $\det(\bar{M}) = \dfrac{(n-1)^2\cdot(2-n)\cdot(n-2)!}{n^{(n-3)}}\cdot t_{n-2}$ which obviously is nonzero when $n>5$.

<u>Case 2): $f(x,y)=x^n+y^{an}$, $n>4$, $a>2$</u>

In this case, $H_0=k[t_2,\ldots,t_{n-1}]$, $\dim \underline{H}_0=n-2$

$$F_0 = x^n+ y^{an} + t_2x^{n-2}y^{2a} + t_3x^{n-3}y^{3a} + \ldots + t_{n-1}xy^{(n-1)a}$$

$$\frac{1}{n}\cdot\frac{\partial F_0}{\partial x}= x^{n-1}+ \frac{(n-2)}{n}t_2x^{n-3}y^{2a} + \frac{n-3}{n}t_3x^{n-4}y^{3a} +\ldots+ \frac{1}{n}t_{n-1}y^{(n-1)a}=0$$

$$\frac{1}{an}\cdot\frac{\partial F_0}{\partial y}=y^{an-1}+ \frac{2}{n}t_2x^{n-2}y^{2a-1}+ \frac{3}{n}t_3x^{n-3}y^{3a-1}+\ldots+ \frac{n-1}{n}t_{n-1}xy^{(n-1)a-1}=0$$

are the fundamental relations in $\mathcal{L}$.

On $U_0$, $B=\{x^{\alpha_1}y^{\alpha_2} | 0\leq\alpha_i\leq n-2, 0\leq\alpha_2\leq an-2\}$ is a local basis for .
There is one monomial in $B$ of weight $\frac{2n-3}{n}$, the one being $x^{n-2}y^{a(n-1)}$. Arguing as before, for $k=0,1,\dots,2n-4$

$$x^k y^{(2n-3-k)a} = \Omega_k \cdot x^{n-2} y^{a(n-1)}$$

where $\Omega_k \in H_0$, $\Omega_k \in (t_2,\dots,t_{n-1})$ for $k\neq n-2$. In fact we easily see that, modulo $(t_2,\dots,t_{n-1})^2$:

$$(3.18)\qquad \begin{aligned} \Omega_{2n-\ell} &= -\frac{(\ell-2)}{n} t_{n+2-\ell}, \quad \ell=3,4,\dots,n \\ \Omega_{n-1} &= 0, \quad \Omega_{n-2} = 1 \\ \Omega_{n-\ell} &= -\frac{(n+2-\ell)}{n} t_{n+2-\ell}, \quad \ell=3,\dots,n \end{aligned}$$

As noted earlier, (3.6) now reads

$$\begin{aligned} v_{(1,0)} &= a_{22} d_{(1,0)} + a_{21} d_{(0,a)} \\ v_{(0,1)} &= a_{11} d_{(0,1)} \end{aligned}$$

that is, $a_{12}$ in the general formula is zero. Furthermore, $\mathfrak{g}(\underline{t})$ is generated by $d_{(0,1)}$, $d_{(1,0)}$, $d_{(0,2)}$, $d_{(2,0)}$.
Let $A=d_{(n-2,a(n-1))}$, $v_A=v_{(n-2,a(n-2))}$. This time, we shall consider the equations

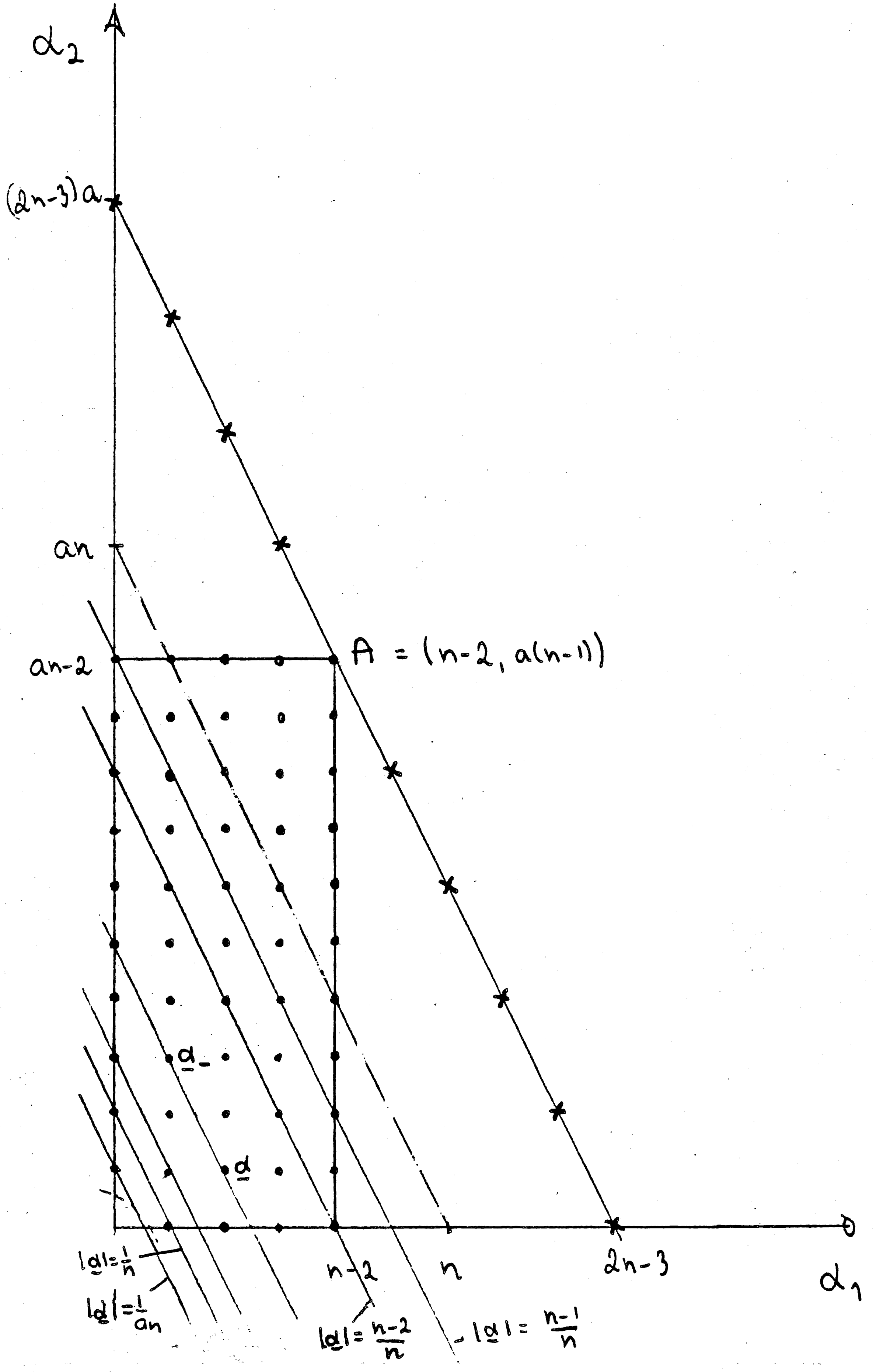
$$D(f_{(n-k,(k-2)a(n-\ell,(\ell-1)a)}) - \lambda_{(n-k,(k-2)a)(n-\ell,(\ell-1)a)} = 0$$

for $k,\ell=2,3,\dots,n$.
The situation is illustrated in Fig. 2.

The computations are similar to the calculations in case 1), only simpler. By (3.5) we get

$$v_A = [(n-2)a_{22} + a(n-1)a_{11} + (n-2)\Omega_{n-3}a_{21}]\cdot A$$

Fig. 2 $\quad f(x,y) = x^n + y^{an} \quad (n=6\ ,\ a=2)$



Using this we find, for $2 \leqslant k, \ell \leqslant n$:

$$n \cdot (D(f_{(n-k,(k-2)a)(n-\ell,(\ell-1)a)}) - \lambda_{(n-k,(k-2)a)(n-\ell,(\ell-1)a)}) =$$

$$\{(\ell+k-2-n)\Omega_{2n-k-\ell} \cdot (a \cdot a_{11} - a_{22})$$

$$+ [(2n-k-\ell)\Omega_{2n-k-\ell} + (2-n)\Omega_{n-3}\Omega_{2n-k-\ell}] \cdot a_{21}$$

$$- \frac{\partial \Omega_{2n-k-\ell}}{\partial t_2} \cdot u_2 - \cdots - \frac{\partial \Omega_{2n-k-\ell}}{\partial t_{n-1}} \cdot u_{n-1}\} \cdot A = 0$$

So each pair $(k, \ell)$ yields a linear equation in $(a \cdot a_{11} - a_{22})$, $a_{21}$, $u_2, u_3, \ldots, u_{n-1}$.

Consider the equations

$$n \cdot (D(f_{(0,(n-2)a)(n-4,3a)}) - \lambda_{(0,(n-2)a)(n-4,3a)}) = 0$$

$$n \cdot (D(f_{(n-2,0)(n-\ell,(\ell+1)a}) - \lambda_{(n-2,0)(n-\ell,(\ell+1)a)}) = 0, \quad \ell = n-1, \ldots, 3, 2$$

$$n \cdot (D(f_{(0,(n-2)a)(n-3,2a)}) - \lambda_{(0,(n-3)a)(n-3,2a)}) = 0$$

We compute the first-order term of the determinant of the coefficient matrix as before. We find

$$\bar{M} = \begin{bmatrix} \frac{(2-n)2}{n} t_{n-2} & 0 & 0 & 0 & \cdots & 0 & \frac{(n-2)}{n} & 0 \\ 0 & (n-1) & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{2(n-2)}{n} t_2 & 0 & \frac{n-2}{n} & 0 & \cdots & 0 & 0 & 0 \\ \frac{3(n-3)}{n} t_3 & 0 & 0 & \frac{n-3}{n} & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & & \\ \frac{(n-2)2}{n} t_{n-2} & 0 & 0 & 0 & \cdots & 0 & \frac{2}{n} & 0 \\ \frac{(1-n)}{n} t_{n-1} & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{n-1}{n} \end{bmatrix}$$

This time, $\det(\bar{M}) = \frac{(2-n)\cdot(n-1)^2\cdot(n-2)!}{n^{n-2}} t_{n-2}$ which is nonzero for $n>4$.

Case 3): $f=x^{an}+y^{bn}$, $n>3$, $2<a<b$, $(a,b)=1$

The third case gives $H_0=k[t_1,\ldots,t_{n-1}]$, $\dim \underline{H}_0=n-1$

$$F_0 = x^{an} + y^{bn} + t_1x^{a(n-1)}y^b + t_2x^{a(n-2)}y^{2b} + \ldots + t_{n-1}x^ay^{b(n-1)}$$

$$\frac{1}{an}\cdot\frac{\partial F_0}{\partial x} = x^{an-1} + \frac{n-1}{n}t_1x^{a(n-1)-1}y^b + \frac{n-2}{n}t_2x^{a(n-2)-1}y^{2b} + \ldots + \\ + \frac{1}{n}t_{n-1}x^{a-1}y^{b(n-1)}$$

$$\frac{1}{bn}\cdot\frac{\partial F_0}{\partial y} = y^{bn-1} + \frac{1}{n}t_1x^{a(n-1)}y^{b-1} + \frac{2}{n}t_2x^{a(n-2)}y^{2b-1} + \ldots + \\ + \frac{n-1}{n}t_{n-1}x^ay^{b(n-1)-1}$$

On $U_0$, the monomial of weight $2-\frac{(2+a)}{an}$ in the chosen basis is $x^{an-2}y^{b(n-1)}$.

Any monomial of this weight may be expressed

$$x^{ak-2}y^{b(2n-k-1)} = \Omega_k\cdot x^{an-2}y^{b(n-1)}$$

Once again, we shall only need to know $\Omega_k$ modulo $(t_2,\ldots,t_{n-1})$, We find

$$\text{(3.19)}\qquad \begin{aligned} \Omega_{n+k} &= -\frac{n-k}{n}t_k\,, \quad 1<k<n-1 \\ \Omega_n &= 1 \\ \Omega_k &= -\frac{k}{n}t_k\,, \quad 1<k<n-1 \end{aligned}$$

Notice that in this case

$$v_{(1,0)} = a_{22} \cdot d_{(1,0)}$$
$$v_{(0,1)} = a_{11} \cdot d_{(0,1)}$$

(Cfr. (3.6)), so that, for $(\alpha_1, \alpha_2) \in I^*$

$$v_{(\alpha_1,\alpha_2)} = (\alpha_2 \cdot a_{11} + \alpha_1 \cdot a_{22}) d_{(\alpha_1,\alpha_2)} .$$

For $k, \ell = 2,3,\ldots,n$ consider the equations

$$n \cdot (D(f_{\underline{\alpha}_k \underline{\beta}_\ell}) - \lambda_{\underline{\alpha}_k \underline{\beta}_\ell}) = 0 \qquad \begin{array}{l} \underline{\alpha}_k = (ak-a-1, (n-k)b) \\ \underline{\beta}_\ell = (a\ell-a-1, (n-\ell+1)b) \end{array}$$

(A look at Fig. 3 might be more enlightening.)

Straightforward checking shows that, in terms of

$a_{11}, a_{22}, u_1, u_2, \ldots, u_{n-1}$ the equations are

$$(k+\ell-n-2) \cdot \Omega_{k+\ell-2} \cdot (a \cdot a_{22} - b \cdot a_{11}) - \frac{\partial \Omega_{k+\ell-2}}{\partial t_1} \cdot u_1 - \ldots - \frac{\partial \Omega_{k+\ell-2}}{\partial t_{n-1}} \cdot u_{n-1} = 0$$

If we choose the equations given by

$k = 2, \quad \ell = n-2$

$k = n, \quad \ell = 3,4,\ldots,n$

$k = 2, \quad \ell = n-1$

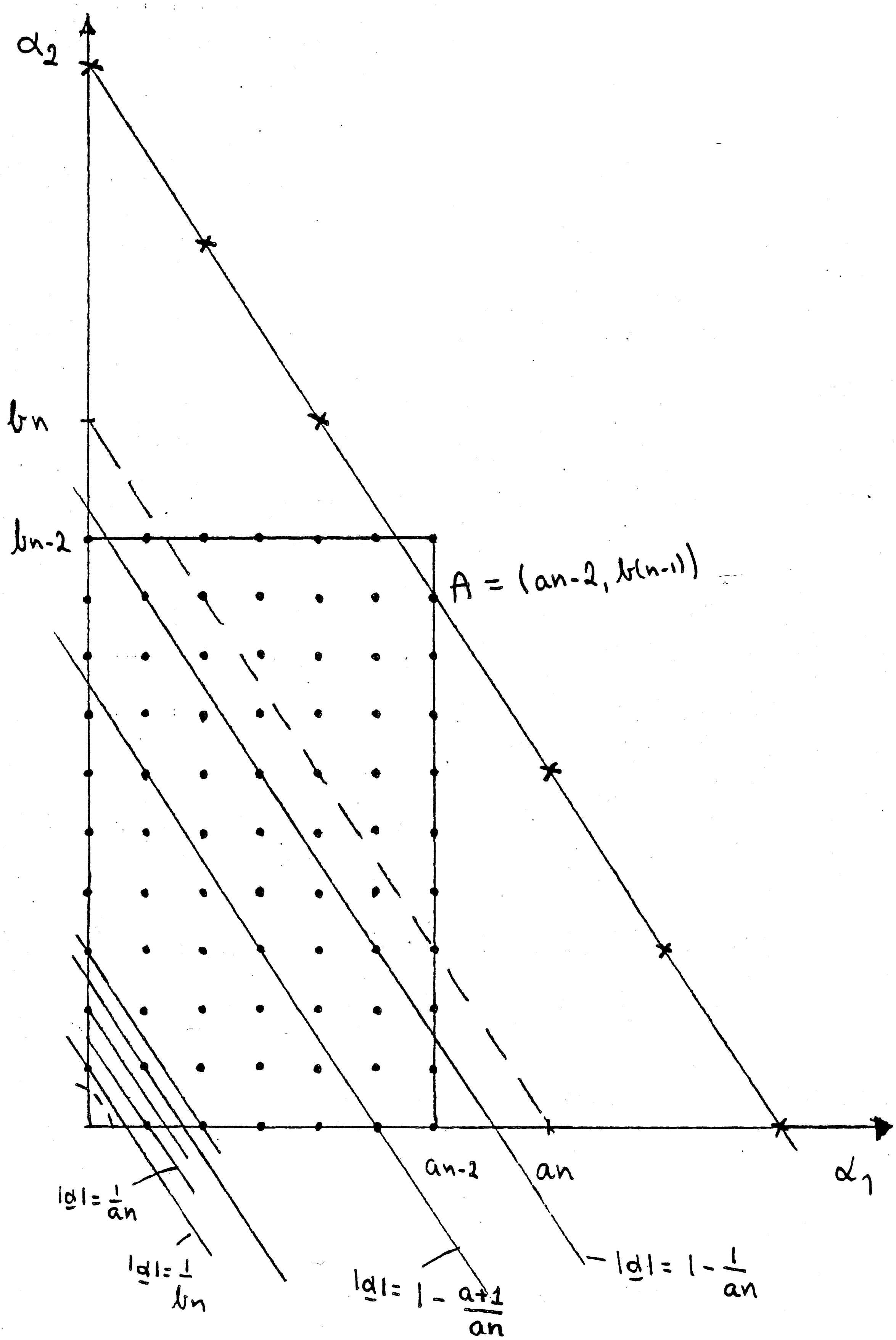
the resulting matrix $\bar{M}$ is

$$\bar{M} = \begin{bmatrix} \frac{2(2-n)}{n} t_{n-2} & 0 & 0 & 0 & \ldots & 0 & \frac{n-2}{n} & 0 \\ \frac{1\cdot(1-n)}{n} t_1 & \frac{n-1}{n} & 0 & 0 & \ldots & 0 & 0 & 0 \\ \frac{2\cdot(2-n)}{n} t_2 & 0 & \frac{n-2}{n} & 0 & \ldots & 0 & 0 & 0 \\ \cdot & & & & & & & \\ \cdot & & & & & & & \\ \frac{(2-n)2}{n} t_{n-2} & 0 & 0 & 0 & \ldots & 0 & \frac{2}{n} & 0 \\ \frac{(n-1)}{n} t_{n-1} & 0 & 0 & 0 & \ldots & 0 & 0 & \frac{n-1}{n} \end{bmatrix}$$

Since $\det(\bar{M}) = \frac{(n-1)\cdot(2-n)\cdot(n-1)!}{n^{n-1}} \cdot t_{n-2}$ $\bar{M}$ obviously has maximal rank for $n>3$.

Q.E.D.

Fig. 3 $f(x,y) = x^{an} + y^{bn}$ (n=4 , a=2 , b=3)



Having proved Theorem (3.2) for dim $H_0 > 2$, we may ask ourselves what happens in dimension 1. We shall not try to answer this question in general, although this might possibly well be done.

However, for each particular quasihomogenous $f$ such that dim $\underline{H}_0 = 1$, one may easily compute the deformation $\mathcal{L}$ and in fact explicitly determine the system of equations

$$D(f_{\underline{\alpha\beta}}) = \lambda_{\underline{\alpha\beta}} \qquad (\underline{\alpha},\underline{\beta} \in I^*)$$

Then it is sheer labour to decide whether these equations may be solved or not. We have examined three examples:

1) $\underline{f(x,y) = x^4+y^4}$ $\qquad F_0(x,y) = x^4+y^4+tx^2y^2$

For $t \neq \pm 2$ $F_0(t)$ defines an isolated singularity, then $L(t)$ is a Lie-algebra of dimension 9. For each $t$, $\{d_{(\alpha_1,\alpha_2)} \mid 0 < \alpha_i < 2\}$ is a basis for $L(t)$.

Now a simple calculation shows that if $B_t = \{x_{(\alpha_1,\alpha_2)} \mid 0 < \alpha_i < 2\}$ is the basis for $L(t)$ given by

i) $x_{\underline{\alpha}} = d_{\underline{\alpha}}$, $\underline{\alpha} = (0,0),(1,0),(0,1)$

ii)) $x_{(2,0)} = d_{(2,0)} + \frac{t}{2} d_{(0,2)}$ , $x_{(0,2)} = d_{(0,2)} + \frac{t}{2} d_{(2,0)}$

iii) $x_{\underline{\alpha}} = (1-(\frac{t}{2})^2) \cdot d_{\underline{\alpha}}$ otherwise

then the structural constants with respect to the basis $B_t$ are independent of $t$. Hence the $L(t)$'s are all isomorphic.

2) $\underline{f(x,y) = x^3+y^6}$ $\qquad F_0(x,y) = x^3+y^6+t \cdot xy^4$

In this case we find that the equations $D(f_{\underline{\alpha\beta}}) = \lambda_{\underline{\alpha\beta}}$ may be solved, except at the origin. In fact the Lie-algebras $\mathcal{C}^1L(f)$ and $\mathcal{C}^1L(t)$, $t\neq 0$, are distinguished by the central ascending series. In each case the center is one-dimensional, whereas the next term is 3-dimensional for $\mathcal{C}^1L(f)$, of dimension 2 for $\mathcal{C}^1L(t)$, $t\neq 0$. Thus the family is only locally constant.

3) $\underline{f(x,y,z) = x^4+y^4+z^3}$ $\qquad F_0(x,y,z) = x^4+y^4+z^3+tx^2y^2$

Computing this example we find that the Lie-algebra $L(F_0)$ is the semidirect product of the constant family $L(x^4+y^4+tx^2y^2)$ and the abelian Lie-algebra $k^9$. It is therefore rather surprising to find that this is a non-constant family of Lie-algebras (that is, the map $\lambda(t)$ is injective for each $t$).

In particular, it is not true that every 1-dimensional family of Lie-algebras is locally constant.

## §4. Maximal tori and Generalized Cartan Matrices to the Lie-algebra $\mathcal{L}^1L(f)$

In this paragraph, we shall determine a maximal torus $T$ on the nilpotent Lie-algebra ${}^1L(f)$, where $f=x_1^{n_1}+\ldots+x_k^{n_k}$. Using this, we may also compute the Generalized Cartan Matrix (G.C.M.) associated to ${}^1L(f)$ for certain plane curve singularities, applying the procedure of [San].

<u>Definition (4.1)</u>: A torus on a Lie-algebra $\mathfrak{g}$ is an abelian subalgebra $T$ of the Lie-algebra $\mathrm{Der}_k\mathfrak{g}$ consisting of semisimple endomorphisms. $T$ is a maximal torus if it is not strictly contained in any other torus.

Whenever $T$ is a torus on $\mathfrak{g}$, the elements of $T$ are simultaneously diagonalizable ($k$ being algebraically closed and $T$ commutative). Hence $\mathfrak{g}$ decomposes into a direct sum of root spaces

$$\mathfrak{g} = \bigoplus_{\beta\in T^*}\mathfrak{g}^\beta$$

where $\mathfrak{g}^\beta=\{x\in \ |t(x)=\beta(t)\cdot x\ \forall\ t\in T\}$

<u>Definition (4.2)</u>: The root system associated to $T$ is

$$R(T) = \{\beta\in T^*|\mathfrak{g}^\beta\neq(0)\}$$

If $T,T'$ are any maximal tori on $\mathfrak{g}$, then by Mostow's theorem there exists $\theta\in\mathrm{Aut}\,\mathfrak{g}$ such that $T'=\theta T\theta^{-1}$ ([Mos]).

<u>Definition (4.3)</u>: A Generalized Cartan Matrix (G.C.M.) is an $\ell\times\ell$ matrix $A$ having integer entries $A_{ij}$, such that

1) $A_{ii} = 2 \qquad i=1,\dots,\ell$

2) $A_{ij} \leq 0 \qquad i,j=1,\dots,\ell, \quad i\neq j$

3) $A_{ij} = 0 \iff A_{ji} = 0 \qquad i,j=1,\dots,\ell$

If $\mathfrak{g}$ is any nilpotent Lie-algebra of finite dimension, then Santharoubane ([San]) provides a method for constructing a G.C.M. associated to $\mathfrak{g}$. If $T$ is a maximal torus on $\mathfrak{g}$, define

$$R^1(T) = \{\beta\in R(T) \mid \mathfrak{g}^{\beta}\not\subseteq \mathcal{C}^1\mathfrak{g}\}$$

We may number the elements of $R(T)$ in such a way that

$$R^1(T) = \{\beta_1,\dots,\beta_{s_1},\beta_{s_1+1},\dots,\beta_{s_1+s_2},\dots,\beta_{s_1+s_2+\dots+s_q}\}$$

where $\{\beta_{s_1+\dots+s_{i-1}+1},\dots,\beta_{s_1+\dots+s_i}\}=R^1(T)_{p_i}$, $i=1,\dots,q$, and

$$R^1(T)_{p_i} = \{\beta\in R^1(T) \mid \dim_K(\mathfrak{g}^{\beta})=p_i\} \ , \qquad p_1<p_2<\dots<p_q$$

Let $s=s_1+s_2+\dots+s_q$, define

$$\ell_i = \dim_K\left(\mathfrak{g}^{\beta_i}/\mathfrak{g}^{\beta_i}\cap\mathcal{C}^1\mathfrak{g}\right) \ , \qquad i=1,\dots s$$

Notice that, since $T(\mathcal{C}^1\mathfrak{g})\subseteq \mathrm{Der}_k\mathfrak{g}(\mathcal{C}^1\mathfrak{g})\subseteq\mathcal{C}^1\mathfrak{g}$, $\mathcal{C}^1\mathfrak{g}$ is generated by root vectors for $T$, hence

$$\ell_1 + \ell_2 + \dots + \ell_s = \ell = \dim_k(\mathfrak{g}/\mathcal{C}^1\mathfrak{g}) \ .$$

Moreover, since each $\ell_i \geq 1$, $\ell\geq s$.

We define a map $f:\{1,\dots,\ell\}\to\{1,\dots,s\}$ by

$$f(i) = k \quad \text{for} \quad \ell_1+\dots+\ell_{k-1}+1\leq i\leq \ell_1+\dots+\ell_k$$

We may now define the G.C.M. $A(T)$ associated to the maximal torus $T$.

<u>Definition (4.4)</u>: For $i,j=1,\ldots,\ell,\ i\neq j$ let

$$A_{ij}(T) = -\mathrm{Min}\{n\in Z_+ \mid (\mathrm{ad}x)^{n+1}(y)=0\ \forall\ x\in \mathfrak{g}^{\beta_{f(i)}},\ \forall\ y\in \mathfrak{g}^{\beta_{f(j)}}\}$$

For $i=1,\ldots,\ell$ let $A_{ii}(T)=2$. Let $A(T)$ be the $\ell\times\ell$ matrix with entries $A_{ij}(T)$, $i,j=1,\ldots,\ell$.

Notice that the numbering of the roots in each subset $R^1(T)_{P_i}$ may be chosen arbitrarily, hence $A(T)$ is not uniquely determined by the torus $T$. However, the dependence of $A(T)$ on the particular ordering chosen is unimportant, in the following sence:

Let $\sum_s^{s_1,\ldots,s_q} \simeq \sum_{s_1}\times\ldots\times\sum_{s_q}$ be the group of permutations of $\{1,\ldots,s\}$ leaving each subset $\{1,\ldots,s_1\},\{s_1+1,\ldots,s_1+s_2\},\ldots$ invariant. Choose any map $g:\{1,\ldots,s\}\to\{1,\ldots,\ell\}$ such that $f\circ g$=identity on $\{1,\ldots,s\}$. For $\delta\in\sum_s^{s_1,\ldots s_q}$ let $\tilde{\delta}=g\circ\delta\circ f$. Then there is an action of $\sum_s^{s_1,\ldots,s_q}$ on the set of $\ell\times\ell$ matrices given by

$$\delta(A) = (A_{\tilde{\delta}_i\tilde{\delta}_j})_{i,j=1,\ldots,\ell}$$

where $A=(A_{ij})_{i,j=1,\ldots,\ell}$, $\delta\in\sum_s^{s_1,\ldots,s_q}$.

If the matrices $A(T),A'(T)$ are constructed as in Definition 4 using two different orderings of the $R^1(T)_{P_i}$, then obviously there exists $\delta\in\sum_s^{s_1,\dots,s_q}$ such that $A'(T)=\delta(A(T))$. Applying Mostow's theorem, Santharoubane proves ([San]):

<u>Theorem</u>: 1) The matrix $A(T)$ is a G.C.M.

2) The $\sum_s^{s_1,\dots,s_q}$ orbit of $A(T)$ is an invariant of $\mathfrak{g}$.

We now return to the solvable Lie-algebras $L(f)$. We shall explicitly determine a maximal torus on $\mathfrak{g}=\mathcal{C}^1L(f)$, using two simple lemmaes:

<u>Lemma (4.5)</u>: Let $T_1$ be any torus on $\mathfrak{g}$, let $T_2$ be any other torus containing $T_1$. Then every root space of $T_1$ is a direct sum of root spaces for $T_2$.

Proof: Any element $\beta_2$ of $T_2^*$ restricts to an element $\beta_1$ of $T_1^*$. Obviously $\mathfrak{g}^{\beta_2}\subseteq\mathfrak{g}^{\beta_1}$, hence the lemma follows.

Q.E.D.

<u>Lemma (4.6)</u>: Let $\{x_1,\dots x_n\}$ be a basis for $\mathfrak{g}$ (as a vector space), $T$ a torus on $\mathfrak{g}$ such that

$$t(x_i) = \beta_i(t)\cdot x_i \quad \forall\ t\in T,\ i=1,\dots,n.$$

If $c_{ij}^k\neq 0$, then $\beta_k=\beta_i+\beta_j$.

Proof: Every $t$ is a derivation of $\mathfrak{g}$, hence

$$\begin{aligned} t([x_i,x_j]) &= [t(x_i),x_j] + [x_i,t(x_j)] \\ &= (\beta_i(t)+\beta_j(t))\cdot[x_i,x_j] = \sum_{k=1}^{n} c_{ij}^k\cdot(\beta_i+\beta_j)(t)\cdot x_k \end{aligned}$$

Also, $t([x_i,x_j])=t(\sum_{k=1}^{n} c_{ij}^k x_k)=\sum_{k=1}^{n} c_{ij}^k\cdot\beta_k(t)\cdot x_k$

Subtracting, we obtain:

$$0=\sum_{k=1}^{n} c_{ij}^k\cdot(\beta_i+\beta_j-\beta_k)(t)\cdot x_k \quad \forall\, t\in T$$

implying $c_{ij}^k\cdot(\beta_i+\beta_j-\beta_k)=0$ in $T^*$, since the $x_i$ are linearly independent.

Q.E.D.

From [La-Pf] we now recall a few simple facts about the Lie-algebra $L(f)$, $f=x_1^{n_1}+\ldots+x_k^{n_k}$:

<u>Lemma (4.7)</u>: Let

$$I=\{\underline{\alpha}=(\alpha_1,\ldots,\alpha_k)\in(\mathbb{Z}_+)^k \mid 0\leq\alpha_i\leq n_i-2, i=1,\ldots,k\},$$

for any $\underline{\alpha}\in I$ let $|\underline{\alpha}|=\sum_{i=1}^{k}\frac{\alpha_i}{n_i}$. Then $\{d_{\underline{\alpha}}\mid\underline{\alpha}\in I\}$ is a basis for $L(f)$. The Lie product is given by

$$[d_{\underline{\alpha}},d_{\underline{\beta}}]=\begin{cases}(|\underline{\beta}|-|\underline{\alpha}|)\cdot d_{(\underline{\alpha}+\underline{\beta})} & \text{if } (\underline{\alpha}+\underline{\beta})\in I\\ 0 & \text{otherwise}\end{cases}$$

Moreover, $\{d_{\underline{\alpha}}\mid\underline{\alpha}\in I\setminus\{\underline{0}\}\}$ is a basis for $\mathcal{C}^1L(f)$.

From now on, let $I^*=I\setminus\{\underline{0}\}$. Notice that $\mathfrak{g}=\mathcal{C}^1L(f)$ is generated as a Lie-algebra by the elements $d_{\underline{\alpha}}$ such that $\sum_{i=1}^{k}\alpha_i<2$. Furthermore, if $n_i\neq n_j$ then

$$d_{(\underline{e}_i+\underline{e}_j)}=\frac{n_i\cdot n_j}{(n_i-n_j)}[d_{\underline{e}_i},d_{\underline{e}_j}]\,,\ \text{where}\ \underline{e}_i=(\underbrace{0,\ldots0,1}_{i},0,\ldots0)$$

Combining these observations with (4.5) and (4.6) we may easily prove

THEOREM (4.8): Let $f(x_1,\dots,x_k)=x_1^{n_1}+\dots+x_k^{n_k}$, $3\leq n_1\leq n_2\leq\dots\leq n_k$. Then there exists a torus $T$ on $\mathfrak{g}=\mathcal{C}^1L(f)$, generated by $t_1,\dots,t_k$, where

$$t_i(d_{\underline{\alpha}}) = \alpha_i\cdot d_{\underline{\alpha}} \quad \forall\ \underline{\alpha}\in I^*,\ i=1,\dots,k$$

$T$ is maximal, except when

$k\leq 4$, $n_1=\dots=n_k=3$

$k=2$, $n_1=n_2=4$

$k=1$, $n_1\leq 6$

in which cases $T$ is contained in a torus of dimension $k+1$. Furthermore, the root system associated to $T$ is

$$R(T) = \{\beta_{\underline{\alpha}}=\alpha_1\cdot t_1^*+\dots\alpha_k\cdot t_k^* \mid \underline{\alpha}\in I^*\}.$$

and for each $\underline{\alpha}$ the root space $\mathfrak{g}^{\beta_{\underline{\alpha}}}$ is one-dimensional, generated by $d_{\underline{\alpha}}$.

Proof: For $\underline{\alpha},\underline{\beta}\in I\setminus\{0\}$, we may write

$$[d_{\underline{\alpha}},d_{\underline{\beta}}] = \sum_{\underline{\gamma}\in I} c^{\underline{\gamma}}_{\underline{\alpha\beta}}\cdot d_{\underline{\gamma}}$$

Obviously, $c^{\underline{\gamma}}_{\underline{\alpha\beta}}\neq 0$ implies $\underline{\gamma}=\underline{a}+\underline{\beta}$, or equivalently

$$\gamma_i = \alpha_i + \beta_i,\quad i=1,\dots,k$$

hence $c^{\underline{\gamma}}_{\underline{\alpha\beta}}\neq 0$ implies $\beta_{\underline{\gamma}}(t_i)=\gamma_i=\alpha_i+\beta_i=(\beta_{\underline{\alpha}}+\beta_{\underline{\beta}})(t_i)$

It follows that $t_1,\ldots,t_k$ are derivations, obviously linearly independent, in other words $T$ is a k-dimensional torus on $\mathfrak{g}$. By definition, for each $t_j$, $\underline{\alpha}\in I^*$, $\beta_{\underline{\alpha}}=\sum_{i=1}^{k}\alpha_i t_i^*$ we have

$$t_j(d_{\underline{\alpha}}) = \alpha_j d_{\underline{\alpha}} = (\sum_{i=1}^{k}\alpha_i t_i^*)(t_j)\cdot d_{\underline{\alpha}} = \beta_{\underline{\alpha}}(t_j)\cdot d_{\underline{\alpha}}$$

Finally we must prove that if $T'$ is any torus containing $T$, then $T'=T$. By (4.5), the root spaces of $T'$ are exactly the root spaces of $T$. Hence let $R(T')=\{\phi_\alpha|\underline{\alpha}\in I\setminus\{\underline{0}\}\}$ be the root system of $T'$. By (4.6), these roots must satisfy the system of linear equations

$$[\phi_{\underline{\gamma}}=\phi_{\underline{\alpha}}+\phi_{\underline{\beta}}] \quad \text{when} \quad c^{\underline{\gamma}}_{\underline{\alpha\beta}}\neq 0$$

We shall prove that this implies that

$$(*) \qquad \phi_{\underline{\alpha}} = \alpha_i\cdot\phi_{\underline{e}_1} + \ldots + \alpha_k\cdot\phi_{\underline{e}_k} \quad \forall\ \underline{\alpha}\in T^*$$

(except for the special cases we have mentioned in the Theorem). Hence the $\phi_{\underline{e}_1},\ldots,\phi_{\underline{e}_k}$ generate $T'^*$, implying

$$\dim_k(T') = \dim_k(T'^*) \leq k = \dim_k T$$

Thus $T'=T$, $\phi_{\underline{e}_i}=t_i^*$ and $\phi_{\underline{\alpha}}=\beta_{\underline{\alpha}}$.

We only need to prove the formula $(*)$ when $\sum_{i=1}^{k}\alpha_i\leq 2$. If $\underline{\gamma}\in I^*$ is such that $\sum_{i=1}^{k}\gamma_i\geq 3$, then we may find $\underline{\alpha},\underline{\beta}\in I^*$, $|\underline{\alpha}|\neq|\underline{\beta}|$ such that $\underline{\gamma}=\underline{\alpha}+\underline{\beta}$. Then $c^{\underline{\gamma}}_{\underline{\alpha\beta}}=(|\underline{\beta}|-|\underline{\alpha}|)\neq 0$, hence

$$\phi_{\underline{\gamma}} = \phi_{\underline{\alpha}} + \phi_{\underline{\beta}}$$

By induction on $\sum_{i=0}^{k} \gamma_i$,

$$\phi_{\underline{\gamma}} = (\alpha_1 \phi_{\underline{e}_1} + \ldots + \alpha_k \phi_{\underline{e}_k}) + (\beta_1 \phi_{\underline{e}_1} + \ldots + \beta_k \phi_{\underline{e}_k})$$
$$= \gamma_1 \phi_{\underline{e}_1} + \ldots + \gamma_k \cdot \phi_{\underline{e}_k}$$

Of course, $(*)$ is trivial when $\sum_{i=1}^{k} a_i = 1$, so we need only consider $(*)$ when $\sum_{i=1}^{k} \alpha_i = 2$. We may write

$$f(x_1,\ldots,x_k) = x_1^{\nu_1} + \ldots + x_{k_1}^{\nu_1} + x_{k_1+1}^{\nu_2} + \ldots + x_{k_2}^{\nu_2} + \ldots + x_k^{\nu_r}$$

where $3 < \nu_1 < \nu_2 < \ldots < \nu_r$. Also, write

$$S = \{1,\ldots,k\} = S_1 \cup S_2 \cup \ldots \cup S_r$$

where $S_i = \{k_{i-1}+1,\ldots,k_i\}$ $(k_0 = 0,\ k_r = k)$.

If $\underline{\alpha} \in I^*$, $\sum_{i=1}^{k} \alpha_i = 2$ then $\underline{\alpha} = \underline{e}_i + \underline{e}_j$ for some $i,j \in S$. We want to prove that

$$(**) \qquad \phi_{(\underline{e}_i + \underline{e}_j)} = \phi_{\underline{e}_i} + \phi_{\underline{e}_j} \qquad i,j \in S$$

Recall that each nonzero structural constant $c^{\underline{\gamma}}_{\underline{\alpha\beta}}$ gives rise to an equation

$$\phi_{\underline{\gamma}} = \phi_{\underline{\alpha}} + \phi_{\underline{\beta}}$$

Furthermore, each $c^{\underline{\gamma}}_{\underline{\alpha\beta}} \neq 0$ corresponds to a triple $(\underline{\alpha},\underline{\beta},\underline{\gamma})$ of elements in $I^*$ such that $|\underline{\alpha}| \neq |\underline{\beta}|$, $\underline{\alpha}+\underline{\beta}=\underline{\gamma}$. We prove $(**)$ in a number of simple steps, the first of which is trivial:

Step 1: If $s\in S_i$, $t\in S_j$, $i\neq j$, then

$$\phi_{(\underline{e}_s+\underline{e}_t)} = \phi_{\underline{e}_s} + \phi_{\underline{e}_t}$$

For the next steps, we shall need to consider pairs of equations

(in $I^*$) $\underline{\gamma} = \underline{\alpha} + \underline{\beta}$ $\quad |\underline{\alpha}|\neq|\underline{\beta}|$

$\underline{\gamma} = \underline{\alpha}' + \underline{\beta}'$ $\quad |\underline{\alpha}'|\neq|\underline{\beta}'|$

then comparing the rigth-hand terms in each pair. The point is of course to use the results of the foregoing steps to prove the formula of the next step, in a recursive fashion. The (simple) calculations are, as usual, left to the reader.

Step 2: If $t\in S_i$, $i>2$, then

$$\phi_{2\underline{e}_t} = 2\cdot\phi_{\underline{e}_t}$$

Step 3: If $s,t\in S_i$, $i>2$, then

$$\phi_{(\underline{e}_s+\underline{e}_t)} = \phi_{\underline{e}_s} + \phi_{\underline{e}_y}$$

Step 4: Suppose $S_2$ is non-empty, $s,t\in S_1$, $(\underline{e}_s+\underline{e}_t)\in I^*$. Then

$$\phi_{(\underline{e}_s+\underline{e}_t)} = \phi_{\underline{e}_s} + \phi_{\underline{e}_t}$$

The only case left is $f(x_1,\ldots,x_k)=x_1^n+\ldots+x_k^n$. In this case we find

Step 5: Let $f=x_1^n+\ldots+x_k^n$. Suppose that either

1) $k>5$, $n>3$

or 2) $k>3$, $n>4$

or 3) $k>2$, $n>5$

or 4) $n>6$

If $(\underline{e}_s+\underline{e}_t)\in I^*$, then

$$\phi_{(\underline{e}_s+\underline{e}_t)} = \phi_{\underline{e}_s} + \phi_{\underline{e}_t}$$

This last step finally proves the Theorem.

Q.E.D.

As an application of Theorem (4.8) we now construct the G.C.M. $A(T)$ for the Lie-algebras associated to various families of plane curve singularities. We also list a few other invariants for the $\mathcal{C}^1L(f)$'s.

If $\mathfrak{g}$ is any nilpotent Lie-algebra of finite dimension, we define

1) the <u>type</u> of $\mathfrak{g}$ : $\dim_k(\mathfrak{g}/\mathcal{C}^1\mathfrak{g})$ = the least number of generators for $\mathfrak{g}$ as a Lie-algebra
2) the <u>nilpotency</u> of $\mathfrak{g}$ : $N=\min\{n\in Z_+ | \mathcal{C}^{n+1}\mathfrak{g}=(0)\}$
3) the <u>Mostow number</u> of $\mathfrak{g}$: $M=\dim_k T$, where $T$ is any maximal torus on $\mathfrak{g}$ (by Mostow's theorem, $M$ is independent of the particular choice of $T$).

<u>INVARIANTS FOR $\mathcal{C}^1L(f)$, $f\in k[x,y]$:</u>

I. <u>$f(x,y)=x^n+y^n$, $n>4$</u>

$$M = \begin{cases} 3 & n=4 \\ 2 & n>5 \end{cases} ; \quad N=2n-6; \quad \text{type} = 5$$

$$A(T) = \begin{bmatrix} 2 & 0 & (4-n) & (3-n) & (2-n) \\ 0 & 2 & (2-n) & (3-n) & (4-n) \\ A & B & 2 & 0 & 0 \\ (3-n) & (3-n) & 0 & 2 & 0 \\ B & A & 0 & 0 & 2 \end{bmatrix} \qquad A = \begin{cases} \frac{4-n}{2} & n \text{ even} \\ \frac{3-n}{2} & n \text{ odd} \end{cases} \qquad B = \begin{cases} \frac{2-n}{2} & n \text{ even} \\ \frac{3-n}{2} & n \text{ odd} \end{cases}$$

II. $f(x,y)=x^m+y^n,\ n>m>4$

$M=2$; $N=m+n-5$; type $= 4$

a)

$A(T) =$

$$\underline{n \text{ odd}}: \begin{bmatrix} 2 & (2-m) & (4-m) & (2-m) \\ (2-n) & 2 & (2-n) & (4-n) \\ A & B & 2 & B \\ \frac{3-n}{2} & \frac{3-n}{2} & \frac{3-n}{2} & 2 \end{bmatrix} \qquad A = \begin{cases} \frac{4-m}{2} & m \text{ even} \\ \frac{3-m}{2} & m \text{ odd} \end{cases} \qquad B = \begin{cases} \frac{2-m}{2} & m \text{ even} \\ \frac{3-m}{2} & m \text{ odd} \end{cases}$$

$$\underline{n \text{ even}}: \begin{bmatrix} 2 & (2-m) & (4-m) & D \\ (2-n) & 2 & (2-n) & (4-n) \\ A & B & 2 & B \\ C & \frac{4-n}{2} & \frac{2-n}{2} & 2 \end{bmatrix} \qquad C = \begin{cases} 0 & n=2m \\ \frac{2-n}{2} & \text{otherwise} \end{cases} \qquad D = \begin{cases} 0 & n=2m \\ 2-m & \text{otherwise} \end{cases}$$

III. $F(x,y)=x^3+y^n,\ n>4$

$M=2$; $N=n-2$; type $= 3$

$$A(T) = \begin{bmatrix} 2 & -1 & C \\ (2-n) & 2 & (4-n) \\ A & B & 2 \end{bmatrix} \qquad A = \begin{cases} 0 & \\ \frac{2-n}{2} & n \text{ even, } n \neq 6 \\ \frac{3-n}{2} & n \text{ odd} \end{cases} \qquad B = \begin{cases} \frac{4-n}{2} & n \text{ even} \\ \frac{3-n}{2} & n \text{ odd} \end{cases} \qquad C = \begin{cases} 0 & n=6 \\ -1 & \text{otherwise} \end{cases}$$

V. $f(x)=x^n,\ n>4$

(Notice that we consider $x^n$ as a polynomial in $k[x]$. Then the Lie-algebra $L(x^n)$ is isomorphic to $L(f)$, where $f$ is the isolated plane curve singularity given by $f(x,y)=x^n+y^2$.)

$$M = \begin{cases} 2 & n<6 \\ 1 & n>7 \end{cases} \; ; \; N=n-4 \; ; \quad \text{type} = 2$$

$$A(T) = \begin{bmatrix} 2 & (4-n) \\ A & 2 \end{bmatrix} \qquad A = \begin{cases} \frac{4-n}{2} & n \text{ even} \\ \frac{3-n}{2} & n \text{ odd} \end{cases}$$

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